HIGH SCHOOL ALGEBRA

COMPLETE COURSE

SLAUGHT & LENNES


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HIGH SCHOOL ALGEBRA

Complete Course

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Boston
ALLYN AND BACON
1908

PREFACE

THE High School Algebra comprises two distinct parts, an Elementary Course, designed solely for the first year, and an Advanced Course, for complete review and further study. The authors recognize the impossibility of combining in a single treatment the qualities suitable for beginners with the more mature point of view necessary for the final study of algebra in secondary schools. This complete book will amply meet the entrance requirements of any college or technical school.

The important features of the Elementary Course are:

1. *Algebra is vitally and persistently connected with arithmetic.* Each principle is first studied in its application to numbers in the Arabic notation. The principles of algebra are thus connected with those already known in arithmetic. Letters are introduced as *abbreviations* for "a number" or "any number." Literal expressions are called number expressions — the vague term "quantity" is not used. In the exercises, Arabic figures are constantly involved in the same manner as letters. Checking by the substitution of particular numbers for letters occurs throughout.

2. *The principles of algebra used in the Elementary Course are enunciated in a small number of short statements — eighteen in all.*

The purpose of these principles is to furnish in simple form a codification of those operations of algebra which are sufficiently different from the ones already familiar in arithmetic to require special emphasis. Such a codification has several important advantages:

By constant reference to these few fundamental statements they become an organic and hence permanent part of the learner's mental equipment.

By their systematic use he is made to realize that the processes of algebra, which seem so multifarious and heterogeneous, are in reality few and simple. Invaluable training is thus afforded in connecting things which are essentially, though not apparently, related.

Such a body of principles furnishes a ready means for the correction of erroneous notions, a constant incitement to effective review, and a definite basis upon which to proceed at each stage of progress.

No attempt is here made at formal demonstration of these principles. It is believed that conviction as to their validity is most effectively brought to a first year pupil by their proper empirical exhibition.

3. *The main purpose of the Elementary Course is the solution of problems rather than the construction of a purely theoretical doctrine as an end in itself.*

While the subject-matter of algebra is developed in a logical sequence around the principles, the attempt is made to connect each principle in a vital manner with the learner's experience by using it in the solution of a large number and great variety of simple problems.

In making the problems the following criteria have been observed:

The subject-matter should be easily within the comprehension of the pupil, and, so far as possible, the problems should be such as one would actually need to solve in passing from

known to unknown data by means of given relations. Such are problems on physical relations in Chapter IV, and those on geometrical relations in Chapters VI and VII.

However, with the pupil's meagre experience it is impossible that all problems should be of this kind. Hence a considerable body of problems has been introduced which involve artificial relations imposed upon numbers. Such problems are of two kinds:

(*a*) Those involving numbers related to concrete things, as, for example, the problems on pages 44 to 46. The data used in this class of problems (with very few exceptions) are such as have a distinct value or real interest in themselves. Every effort has been made to have the data entirely trustworthy.

(*b*) Those involving numbers not related to concrete things, as, for example, problems 12 to 18, page 43. Such problems are used as the basis for the development of formulas, pages 110 to 114, Chapter IV.

The utmost care has been taken in grading the problems, both as to difficulty and with reference to the principles upon which the solutions depend.

4. *The order of topics and the inclusion and exclusion of subject-matter have been determined by the main purpose of the Elementary Course as just stated.*

The equation, therefore, as the instrument for the solution of problems, occupies the leading place. New topics are introduced only as they are needed in extending the use of the equation. Thus the subject of factoring is introduced when quadratic equations are to be solved, and is immediately used for that purpose in special cases.

Long division of polynomials is first found necessary as an introduction to square root, which, together with radicals, is essential to the solution of the general quadratic equation.

The postponing to the Advanced Course of various topics, such as Highest Common Factor by the long division process,

has permitted the introduction of a much greater number and larger variety of problems than would otherwise be possible without diminishing the number of drill exercises.

For example, instead of an extended and purely theoretical discussion of radicals, only so much is given as is needed in treating simple quadratic equations; and this is immediately applied to the solution of a body of interesting problems involving such concrete geometrical relations as are freely used in modern grammar school arithmetics.

This larger space allotted to problems makes it possible to pass by slow gradation from extremely simple to more complicated cases which would otherwise be too difficult, and thus to secure the facility and power in interpreting and solving problems which is demanded in physics and other sciences.

The development of the Advanced Course is based upon the following important considerations:

1. The pupil has had a one year's course in algebra, involving constant application of its elementary processes to the solution of concrete problems. This has invested the processes themselves with an interest which now makes them a proper object of study for their own sake.

2. The pupil has, moreover, developed in intellectual maturity and is, therefore, able to comprehend processes of reasoning with abstract numbers which were entirely beyond his reach in the first year's course.

In consequence of these considerations, the treatment throughout is from a more mature point of view than in the Elementary Course. Relatively greater space and emphasis are given to the manipulation of standard algebraic forms, such as the student is likely to meet in later work in mathematics and physics, and especially such as were too complicated for the Elementary Course.

Attention is called to the following special features of the book.

In the Elementary Course:

The simple and scientific treatment of the solution of the equation as enunciated in Principle VIII, page 36.

The numerous illustrative solutions which have been given in full for the purpose of exhibiting the best methods of attack and the most effective arrangement of the work.

The development of negative numbers from concrete relations, immediately followed by a large number of problems showing their practical use and interpretation.

The introduction of graphs at the *beginning* of the chapter on simultaneous equations, thus making the graph the basis of the study of simultaneous equations.

The notions of a formula and of substitution in a formula are developed in connection with subjects already familiar, such as interest, areas, volumes, etc. (See page 101.) These notions are then applied to the study of other subjects. (See page 115.)

Literal equations are in each case introduced as a generalization of a series of concrete problems immediately preceding. (See page 111.) Such equations are then to be solved for each letter involved. (See page 127.)

Problems involving physical relations are in each case introduced by means of a series of carefully graded problems leading up to the general case. (See page 120.)

If in any case the problems are found to be too numerous, the later sections of Chapter IV may be omitted or postponed until after graphs and simultaneous equations have been studied. Many of the exercises are simple enough to be read in class as mental drill exercises.

Review questions and exercises are given at appropriate intervals throughout the book.

In the Advanced Course:

The clear and simple treatment of equivalent equations in Chapter III.

The discussion by formula, as well as by graph, of inconsistent and dependent systems of linear equations.

The unusually complete treatment of factoring and the clear and simple exposition of the general process of finding the Highest Common Factor, in Chapter V.

The careful discrimination in stating and applying the theorems on powers and roots in Chapter VI.

The unique treatment of quadratic equations in Chapter VII, giving a lucid exposition in concrete and graphical form of distinct, coincident, and imaginary roots.

The concise treatment of radical expressions in Chapter X, and especially — an innovation much needed in this connection — the rich collection of problems, in the solution of which radicals are applied.

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CHICAGO AND BOSTON,
June, 1908.

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Part I

ELEMENTARY COURSE

HIGH SCHOOL ALGEBRA

ELEMENTARY COURSE

CHAPTER I

INTRODUCTION TO THE EQUATION

ADDITION OF NUMBERS HAVING A COMMON FACTOR

1. **Algebra** like arithmetic deals with **numbers**. The numbers of algebra include those used in arithmetic, and also other numbers which are defined as need arises for them. The symbols employed to represent numbers, and the principles used in operating upon them, are introduced by means of illustrative problems as we proceed.

2. **Illustrative Problem.** The shortest railway route from Chicago to New York is 912 miles. How long does it take a train averaging 38 miles an hour to make the journey?

Solution in words. The product of the average number of miles per hour and the required number of hours equals the whole distance traveled. That is, 38 multiplied by "the required number of hours" equals 912. Hence "the required number of hours" is one thirty-eighth of 912, or 24.

Solution using abbreviations. If, instead of the expression "the required number of hours," we use the word "time," or simply the abbreviation t, the solution may be written:

$$38 \times t = 912.$$

Hence $$t = 912 \div 38 = 24.$$

Illustrative Problem. If in the above problem a train makes the journey in 18 hours, find the average number of miles per hour.

Solution. In words we have, as before, 18 multiplied by "the average number of miles per hour" equals 912. Hence "the average number of miles per hour" equals one-eighteenth of 912, or $50\frac{2}{3}$.

Using for the expression "the average number of miles per hour," the word "rate," or simply the abbreviation r, the solution reads:

$$18 \times r = 912.$$
$$r = 912 \div 18 = 50\tfrac{2}{3}.$$

It should be clearly understood that t and r represent *numbers* whose values are unknown at the outset, but which become known at the conclusion of the solution.

3. **Abbreviations representing numbers and operations** such as are used in these problems occur constantly in algebra. In fact, the systematic use of such abbreviations is one of the chief distinctions between arithmetic and algebra.

4. The signs $+$, $-$, $\times$, $\div$, and $=$ are used in algebra with the same meaning that they have in arithmetic. However, instead of $\times$ a point written above the line is often used. Thus $2 \cdot 3$ means 2×3.

The product of two numbers represented by letters is generally indicated by writing the letters consecutively with no sign between them.

Thus rt means the same as $r \cdot t$ or $r \times t$. For example, in such problems as those just given we would write $rt = d$, meaning "the number of miles per hour multiplied by the number of hours equals the distance or number of miles traveled."

Similarly $2\,r$ means $2 \cdot r$, but (as in arithmetic) 25 means $20 + 5$, not $2 \cdot 5$.

A fraction is often used to indicate division

Thus $$\frac{2}{3} = 2 \div 3;\ \frac{r}{t} = r \div t.$$

PROBLEMS

In the same manner as above solve the following problems, first by writing out the solution in words and then by using such abbreviations as may be found convenient.

1. Five times a certain number equals 80. What is the number? Use n for the number.

2. Twelve times a number equals 132. What is the number?

3. A tank holds 750 gallons. How long will it take a pipe discharging 15 gallons per minute to fill the tank?

4. The cost of paving a block on a certain street was \$7 per front foot. How long was the block if the total cost was \$4620?

5. A city lot sold for \$7500. What was the frontage if the selling price was \$225 per front foot?

6. An encyclopedia contains 18,000 pages. How many volumes are there if they average 750 pages to the volume?

7. What is the cost of fencing per rod if it requires \$256 to fence a quarter section of land?

8. A square court 60 feet on a side is to be paved with square tiles. How many square feet in each tile if the pavement requires 1600 tiles?

9. An excavation for a building is to be 130 feet long, 80 feet wide, and 9 feet deep. If 900 cubic feet are removed each day, how long will it require to complete it?

10. A railway embankment which contains 48,900 cubic yards of earth was completed in 163 days. At what rate was it filled in?

11. Taking the length of the earth's orbit as 584 million miles, find how far the earth travels in one day; also in one hour.

5. Definition. If a number is the product of two or more numbers, then these numbers are called **factors** of the given number.

E.g. The integral factors of 10 are 1 and 10 or 2 and 5. $2\frac{1}{2}$ and 4 are also factors of 10.

6. Illustrative Problem. Divide the number 84 into three parts such that the second part is five times the first and the third part is eight times the first.

Solution. Using the abbreviation p for "the first part of the number," we have

$$1 \cdot p \text{ or } p = \text{the first part,}$$
$$5p = \text{the second part,}$$
$$8p = \text{the third part.}$$

Since the sum of the three parts is 84,

$$p + 5p + 8p = 84.$$

To complete the solution of this problem it is necessary to find the sum of the numbers p, $5p$, and $8p$ without knowing what number is represented by p itself. If we suppose this sum to be $14p$, then

$$14p = 84$$

and

$$p = 84 \div 14 = 6.$$

This supposition leads to the correct result since, if $p = 6$, then $p + 5p + 8p = 6 + 5 \cdot 6 + 8 \cdot 6 = 6 + 30 + 48 = 84$. Hence the three parts into which 84 is divided are 6, 30, and 48.

7. The process here used for adding p, $5p$, and $8p$ is a *new method of adding* which is of very great importance in algebra. This method is further exhibited by the following examples:

(1) To add 18, 42, 54, and 30, we first factor these numbers so as to show the common factor 6 and then add the remaining factors 3, 7, 9, and 5, and multiply this sum by the common factor 6. The result is $24 \cdot 6$, or 144. The work may be arranged as follows:

$$\begin{array}{r@{\;=\;}l}
18 & 3 \cdot 6 \\
42 & 7 \cdot 6 \\
54 & 9 \cdot 6 \\
30 & 5 \cdot 6 \\
\hline
144 & 24 \cdot 6
\end{array}$$

Similarly (2)

$$\begin{array}{r@{\;=\;}r@{\;=\;}r@{\;=\;}r}
16 & 1 \cdot 16 & 2 \cdot 8 & 4 \cdot 4 \\
64 & 4 \cdot 16 & 8 \cdot 8 & 16 \cdot 4 \\
32 & 2 \cdot 16 & 4 \cdot 8 & 8 \cdot 4 \\
48 & 3 \cdot 16 & 6 \cdot 8 & 12 \cdot 4 \\
\hline
160 & 10 \cdot 16 & 20 \cdot 8 & 40 \cdot 4
\end{array}$$

From example (2) we see that in case the numbers to be added have two or more common factors it does not matter

which one is selected. If each number is separated into *two* factors one of which is a common factor, then it is evident that the sum in any case is found by multiplying the common factor by the sum of the other factors.

In the above manner add the following sets of numbers and show in each case by adding in the ordinary way that the result is correct:

14	36	32	17	39	32
28	72	64	34	52	48
35	48	128	68	78	40
70	12	256	85	91	72

8. **Definition.** If a number is the product of two factors, then either of these factors is called the **coefficient** of the other in that number.

E.g. In $2 \cdot 3$, 2 is the coefficient of 3, and 3 is the coefficient of 2. In $9\,rt$, 9 is the coefficient of rt, r is the coefficient of $9\,t$, and t is the coefficient of $9\,r$. In such expressions as $9\,rt$ the factor represented by Arabic figures is usually regarded as the coefficient.

The preceding examples illustrate the following principle:

9. **Principle I.** *To add numbers having a common factor, add the coefficients of the common factor and multiply the sum by the common factor.*

If n represents some number, then $4\,n$, in the same discussion, means four times that number, and $5\,n$ means five times that number. Hence, by Principle I, $4\,n + 5\,n = 9\,n$, which is true no matter what number is represented by n.

By means of this principle perform the following additions, understanding that each letter represents some number:

1. $8x + 7\,x + 16\,x + 2\,x$.
2. $13\,n + 8\,n + 7\,n + 9\,n$.
3. $3\,a + a + 2\,a + 6\,a$.
4. $7\,b + 8\,b + b + 6\,b$.
5. $8\,t + 7\,t + 5\,t$.
6. $3\,r + 5\,r + 11\,r$.

Show the correctness of the result in each of the above by letting $x=2$, $n=1$, $a=4$, $b=3$, $t=6$, and $r=7$. Try also other values for the letters.

Such a test for the correctness of an operation is called a **check**.

10. **Definitions.** Combinations of Arabic figures or letters, or both, by means of the signs of operation, $+$, $-$, etc., are called **number expressions.**

E.g. 38, $18r$, $p+5p+8p$, are number expressions.

Two number expressions representing the same number, when connected by the sign $=$, form an **equality.**

The expressions thus connected are called the **members** of the equality and are distinguished as the **right and left** members.

Equalities, such as $8t+7t=15t$, in which the letters may be any numbers whatever, are called **identities.** Equalities of the type $p+5p+8p=84$ are called **equations.** (See §§ 29–34.)

EXERCISES

1. Add $5t$, $11t$, $20t$, and $47t$. Check the result by letting $t=3$; also $t=50$, and $t=150$.

2. Add $7r$, $23r$, $28r$, $52r$, and $117r$. Check the result for $r=11$, $r=20$, and $r=1$.

3. Add $3rt$, $7rt$, $65rt$, and $16rt$. Check for $r=1$, $t=2$.

4. Add $1\frac{1}{2}n$, $2\frac{2}{3}n$, $3\frac{1}{6}n$, and check for $n=6$.

5. Add 66, 88, 99, and 121 by Principle I.

6. Add 144, 96, 120, and $50 \cdot 12$ by Principle I.

7. Add $5 \cdot 40$, $8 \cdot 60$, and $6 \cdot 20$ by Principle I.

8. Add $5ax$, $3ax$, and $7ax$. Check for $a=2$, $x=4$.

9. Add $7ax$, $3bx$, and $12cx$.

In this case the common factor is x. Hence using Principle I, $7ax+3bx+12cx=(7a+3b+12c)x$. The parenthesis here indicates that the numbers represented by x and the expression $7a+3b+12c$ are to be multiplied.

SOLUTION OF PROBLEMS

11. One great object in the study of algebra is to **simplify the solution of problems.** This is done by using letters to represent the unknown numbers, by stating the problem in the form of an equation, and by arranging the successive steps of the solution in an orderly manner.

Skill in translating problems into equations depends upon attention to the following points:

(1) *Read and understand* clearly the statement of the problem, as it is given in words.

(2) *Select the unknown number,* and represent it by a suitable letter, say the initial letter of a word which will keep its meaning in mind. If there are more unknown numbers than one, try to express the others in terms of the one first selected.

(3) Find two number expressions which, according to the problem, represent the same number, and set them equal to each other, *thus forming an equation.*

These steps are exhibited in the following solution:

A tree 108 feet high was broken off by the wind so that the part left above the first branch was three times as long as the part broken off, and the part below the first branch was twice as long as the part broken off. How long was the part broken off?

Solution. Let b represent the number of feet broken off.

Then $3b$ is the number of feet left above the first branch,

and $2b$ is the number of feet below the first branch.

Hence, $b + 3b + 2b$ and 108 are number expressions, each representing the total height of the tree.

Therefore $$b + 3b + 2b = 108. \quad (1)$$

By Principle I, $$6b = 108. \quad (2)$$

Then $$b = \text{one sixth of } 108, \text{ or } 18. \quad (3)$$

Hence, the part broken off was 18 feet long.

Equation (3) is derived from (2) by dividing both members by 6.

PROBLEMS

1. The greater of two numbers is 5 times the less, and their sum is 180. What are the numbers?

2. A number increased by twice itself, 4 times itself, and 6 times itself, becomes 429. What is the number?

3. A father is 3 times as old as his son, and the sum of their ages is 48 years. How old is each?

4. In a company there are 39 persons. The number of children is twice the number of grown people. How many are there of each?

5. A and B receive $45 for doing a certain piece of work. If A gets 4 times as much as B, how much does each receive?

6. The population of Tokio is twice that of Canton, and the sum of their populations is 2,700,000. How many inhabitants in each city?

7. Find two consecutive integers whose sum is 133.

8. The area of Louisiana is (nearly) 4 times that of Maryland, and the sum of their areas is 60,930 square miles. Find the (approximate) area of each state.

9. The horse-power of a certain steam yacht is 12 times that of a motor boat. The sum of their horse-powers is 195. Find the horse-power of each.

10. There are three circles on the blackboard. The circumference of the second is 5 times that of the first, and the circumference of the third is 10 times that of the first. The sum of their circumferences is 16 feet. Find the circumference of each.

11. At a football game there were 2000 persons. The number of women was 3 times the number of children, and the number of men was 6 times the number of children. How many men, women, and children were there?

12. The population of Portland, Oregon (estimate of the Census Bureau, 1904), was twice that of Dallas, Texas, and the population of Toledo was 3 times that of Dallas. The three cities together had 300 thousand inhabitants. How many were there in each city?

Let n = number of thousands of inhabitants in Dallas.

Then $2n$ = number of thousands of inhabitants in Portland,
and $3n$ = number of thousands of inhabitants in Toledo.

Hence $$n + 2n + 3n = 300.$$

13. It is twice as far from New York to Syracuse as from New York to Albany, and it is 4 times as far from New York to Cleveland as from New York to Albany. The sum of the three distances is 1015 miles. Find each distance.

14. In Maryland there were (census of 1900) 4 times as many whites as negroes. The total population was 1185 thousand. How many of each were there?

If n equals the number of thousands of negroes, then the equation is $n + 4n = 1185$.

SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

12. Numbers having a common factor may be subtracted in a manner similar to the process exhibited under Principle I.

Thus, from	$64 = 8 \cdot 8$	From	$84 = 12 \cdot 7$	From	$17n$
subtract	$48 = 6 \cdot 8$	subtract	$49 = 7 \cdot 7$	subtract	$6n$
Remainder	$16 = 2 \cdot 8$		$35 = 5 \cdot 7$		$11n$

In like manner perform the following subtractions:

1. $9 \cdot 7 - 3 \cdot 7$.	**5.** $6 \cdot 99 - 5 \cdot 99$.	**9.** $6n - 2n$.
2. $10 \cdot 4 - 6 \cdot 4$.	**6.** $20 \cdot 19 - 13 \cdot 19$.	**10.** $6 \cdot 50 - 2 \cdot 50$.
3. $8 \cdot 8 - 2 \cdot 8$.	**7.** $8a - 3a$.	**11.** $10b - 4b$.
4. $5 \cdot 11 - 3 \cdot 11$.	**8.** $8 \cdot 5 - 3 \cdot 5$.	**12.** $7a - 4a$.

These examples illustrate the following principle:

13. Principle II. *To find the difference of two numbers having a common factor, subtract the coefficients of the common factor and multiply the result by the common factor.*

Illustrative Problem. If thirteen times a certain number diminished by eight times the number equals 75, what is the number?

Solution. Let n represent the required number.

Then $13n - 8n$ and 75 are expressions representing the same number.

Hence, $13n - 8n = 75.$

By Principle II, $5n = 75.$

Dividing each member by 5, $n = 15$, the required number.

Check. $13 \cdot 15 - 8 \cdot 15 = 195 - 120 = 75.$

EXERCISES AND PROBLEMS

1. By means of Principle II subtract 72 from 160; 50 from 300; 39 from 78; 34 from 85; 58 from 174; and 69 from 161.

2. Subtract $109 \cdot 87$ from $209 \cdot 87$ by Principle II. Check by first finding the products and then subtracting as in arithmetic.

Perform the following indicated operations and check those in which letters are involved by substituting convenient numbers:

3. $68t - 11t.$
4. $15n + 25n - 18n.$
5. $70x - 15x + 7x - 23x.$
6. $18 \cdot 7 - 3 \cdot 7 - 2 \cdot 7 + 6 \cdot 7.$
7. $3 \cdot 4n + 5 \cdot 4n + 11 \cdot 4n - 7 \cdot 4n.$
8. $13rt + 16rt + 3rt - 20rt.$
9. $144 - 96 + 50 \cdot 12 - 20 \cdot 12.$
10. $11ax - 3ax + 4ax.$

11. $11\,ax - 3\,bx + 4\,cx$.
12. $11 \cdot 9 - 6 \cdot 9 + 3 \cdot 9$.
13. $20\,n - 6\,n + 2\,n$.
14. $an - bn + cn$.
15. $5\,t + 20\,t - 3\,t$.
16. $8\,s - 3s + 20\,s$.
17. $6\,a - 4\,a + 3\,a - 2\,a$.
18. $11\,rs - 2\,rs + 4\,rs$.
19. $ar + br - cr$.
20. $3\,ry - 2\,sy - ty$.
21. $11 \cdot 17 + 47 \cdot 17 - 8 \cdot 17$.
22. $axy + bxy - 3\,xy$.
23. $3\,abc + 7\,abc - 2\,abc$.
24. $7 \cdot 5\,x - 3 \cdot 5\,x + 8 \cdot 5\,x$.
25. $a \cdot 2\,r + b \cdot 2\,r - c \cdot 2\,r$.
26. $2\,ar + 2\,br - 2\,cr$.

27. Four times a certain number plus 3 times the number minus 5 times the number equals 48. What is the number?

28. One number is 4 times another, and their difference is 9. What are the numbers?

29. Find a number such that when 4 times the number is subtracted from 12 times the number the remainder is 496.

30. The population of Ohio (1901) was twice that of Wisconsin. The difference of their populations was 2100 thousand. Find the population of each state.

31. The population of Illinois in 1903 was 5 times as great as that of West Virginia. The difference between their populations was 4080 thousand. What was the population of each?

32. There are three numbers such that the second is 11 times the first and the third is 27 times the first. The difference between the second and the third is 64. Find the numbers.

33. A cubic foot of asphaltum is twice as heavy as a cubic foot of light anthracite coal. Seven cubic feet of coal weigh 390 pounds more than 1 cubic foot of asphaltum. Find the weight per cubic foot of each.

34. Thirty-nine times a certain number, plus 19 times the number, minus 56 times the number, plus 22 times the number, equals 12. Find the number.

MULTIPLICATION OF A PRODUCT

14. Illustrative Problem. Three men, A, B, and C, invest together $33,000. B puts in twice as much as A, and C 4 times as much as B. How much does each invest?

Solution. Let d represent the number of dollars invested by A. Then $2d$ represents B's investment, and $4(2d)$ represents C's investment.

Hence, $$d + 2d + 4(2d) = 33000.$$

To complete the solution of this problem it is necessary to multiply $2d$ by 4 without knowing what number is represented by d. If we suppose the product to be $8d$,

then $$d + 2d + 8d = 33000.$$

By Principle I, $$11d = 33000.$$

Dividing each member by 11, $d = 3000$, A's investment;

$2d = 6000$, B's investment;

$4 \cdot 2d = 24000$, C's investment.

The supposition that $4(2d) = 8d$ is justified by the fact that the numbers thus found satisfy the conditions of the problem.

That is, $d + 2d + 4(2d) = 3000 + 6000 + 24000 = 33000$.

15. The process here used for multiplying $2d$ by 4 is of great importance in algebra.

The following examples further exhibit this *new method of multiplying:*

1. $4(3 \cdot 5) = 4 \cdot 15 = 60$
Also $4(3 \cdot 5) = 12 \cdot 5 = 60$
And $4(3 \cdot 5) = 3 \cdot 20 = 60$

2. $2(3 \cdot 4 \cdot 5) = 2 \cdot 60 = 120$
Also $2(3 \cdot 4 \cdot 5) = 6 \cdot 4 \cdot 5 = 120$
And $2(3 \cdot 4 \cdot 5) = 3 \cdot 8 \cdot 5 = 120$
And $2(3 \cdot 4 \cdot 5) = 3 \cdot 4 \cdot 10 = 120$

In like manner find the following products in two or more ways:

3. $6(2 \cdot 3)$.
4. $4(25 \cdot 99)$.
5. $2(5 \cdot 149)$.
6. $4(19 \cdot 5)$.
7. $20(5a \cdot 4)$.
8. $16(4x \cdot 7)$.
9. $8(4x \cdot 3)$.
10. $9(xy \cdot 5)$.

These examples illustrate the following principle:

16. **Principle III.** *To multiply the product of several factors by a given number, multiply any one of the factors by that number, leaving the others unchanged.*

EXERCISES AND PROBLEMS

Multiply as many as possible of the following in two or more ways. Check where letters are involved.

1. $7(3 \cdot 4 \cdot 5)$.
2. $8(7 \cdot 2 \cdot 3)$.
3. $9(2 \cdot 3 \cdot 4)$.
4. $5(2ab)$.
5. $3(5xy)$.
6. $12(8 \cdot 4 \cdot 20)$.
7. $4(19 \cdot 25)$.
8. $5(17 \cdot 20 \cdot 3)$.
9. $15(7ab)$.
10. $3(4mn)$.
11. $5(abc)$.
12. $7(2xy)$.
13. $2(rs \cdot 16)$.
14. $5(xy \cdot 5z)$.
15. $7(3 \cdot 4ab)$.
16. $9(a \cdot 5 \cdot xy)$.
17. $4(125 \cdot 17)$.
18. $40(25 \cdot 29)$.

19. There are three numbers whose sum is 80. The second is 3 times the first and the third is twice the second. What are the numbers?

20. There are three numbers such that the second is 8 times the first and the third is 3 times the second. If the second is subtracted from the third the remainder is 48. Find the numbers.

21. The population of Bridgeport, Connecticut, is twice that of Butte, Montana. Three times the population of Bridgeport plus twice that of Butte equals 320 thousand. Find the population of each city.

22. It is 4 times as far from New York City to Cincinnati as from New York to Baltimore. Twice the distance from New York to Cincinnati minus 5 times that from New York to Baltimore equals 567 miles. How far is it from New York to each of the other cities?

23. The population of Hartford, Connecticut, is 3 times that of Oshkosh, Wisconsin. Four times the population of Hartford plus 5 times that of Oshkosh equals 510 thousand. Find the population of each city.

24. One cubic inch of emery weighs twice as much as 1 cubic inch of ivory. The combined weight of 10 cubic inches of each substance is 2.1 pounds. Find the weight per cubic inch of each.

25. It is twice as far from Boston to Quebec as from Boston to Albany and 3 times as far from Boston to Jacksonville, Florida, as from Boston to Quebec. How far is it from Boston to each of the other three cities, the sum of the distances being 1818 miles?

26. A cubic inch of porcelain china is twice as heavy as a cubic inch of ebony, and a cubic inch of rolled zinc is 3 times as heavy as a cubic inch of porcelain. The combined weight of 1 cubic inch of each substance is .387 pounds. Find the weight per cubic inch of each.

MULTIPLICATION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

17. **Illustrative Problem.** The length and width of a rectangle together equal 58 inches. If the width were 5 inches greater, the length of the rectangle would then be twice its width. Find its dimensions.

Solution. Let w represent the number of inches in the width. If it were 5 inches wider, it would then be $w + 5$ inches. Hence the length is $2(w+5)$, the parenthesis indicating that the sum of w and 5 is to be found and the result multiplied by 2.

Hence the sum of the length and width is

$$w + 2(w + 5) = 58.$$

The solution of this problem involves the multiplication of the number $(w + 5)$ by 2 without knowing what number is represented by w.

If we suppose that $2(w+5)=2w+10$,

then we have $\quad w+2w+10=58. \quad$ (1)

By Principle I, $\quad 3w+10=58. \quad$ (2)

Hence $\quad 3w=48$, since 58 is 10 more than $3w$, (3)

and $\quad w=16$, the width. (4)

Then $\quad 58-16=42$, the length.

The correctness of the above process is shown by the fact that the numbers 42 and 16 fulfill the conditions of the problem;

that is, $\quad 2(16+5)=2\cdot 21=42$,

and $\quad 16+42=58$.

Equation (3) is derived from (2) by subtracting 10 from each member.

18. The multiplication of the number $(w+5)$ by 2 without knowing in advance what number is represented by w is a *new method of multiplying* which is constantly used in algebra. The following examples further exhibit this method:

(1) $\quad 4(2+7)=4\cdot 9=36$,

or $\quad 4(2+7)=4\cdot 2+4\cdot 7=8+28=36$.

(2) $\quad 3(3+8+9)=3\cdot 20=60$,

or $\quad 3(3+8+9)=3\cdot 3+3\cdot 8+3\cdot 9=9+24+27=60$.

It is thus seen that in each case the same result is obtained whether we first add the numbers in the parenthesis and then multiply the sum or first multiply the numbers in the parenthesis one by one and then add the products.

Multiply each of the following in two ways where possible:

1. $3(2+7)$.	**5.** $3(a+6)$.	**9.** $x(3+7+10)$.
2. $5(3+4+5)$.	**6.** $11(h+k)$.	**10.** $15(x+y+z)$.
3. $8(5+9+7)$.	**7.** $4(5a+7b+c)$.	**11.** $20(m+n+p)$.
4. $7(6+11+9+9)$.	**8.** $a(5+4+7)$.	**12.** $9(2r+3s+t)$.

19. In a manner similar to the above the difference of two numbers represented by Arabic figures may be multiplied by a given number in either of two ways.

E.g. $6(8-3)=6\cdot 5=30,$

or $6(8-3)=6\cdot 8-6\cdot 3=48-18=30.$

The same result is obtained whether we first perform the subtraction indicated in the parenthesis and then multiply the difference, or first multiply the numbers separately and then subtract the products. In the case of numbers represented by *letters* evidently the second process only is available.

E.g. $6(r-t)=6r-6t.$

Perform as many as possible of the following multiplications in two ways:

1. $7(9-2)$.	4. $17(18-11)$.	7. $5(x-1)$.	10. $m(r-s)$.
2. $12(17-7)$.	5. $9(a-2)$.	8. $3(y-2)$.	11. $x(y-z)$.
3. $5(12-8)$.	6. $8(h-4)$.	9. $a(c-d)$.	12. $t(u-v)$.

The second of the above methods is needed in problems like the following:

Illustrative Problem. The rate of an express train plus that of a freight is 70 miles per hour. If the rate of the freight were 7 miles less, the express would be going twice as fast as the freight. Find the rate of each.

Solution. Suppose the rate of the freight is now r miles per hour.

Then $r-7$ is the supposed rate of the freight,

and $2(r-7)$ is the present rate of the express.

Hence $r+2(r-7)=70$, the sum of the rates, (1)

and $r+2r-14=70.$ (2)

By Principle I, $3r-14=70.$ (3)

Then $3r = 84$, since 70 is 14 less than $3r$. (4)

Hence $r = 28$, the rate of the freight,

and $2(28 - 7) = 2 \cdot 21 = 42$, the rate of the express.

Check. $42 + 28 = 70$.

Equation (4) is derived from (3) by adding 14 to both members and observing that $14 - 14 = 0$.

The foregoing examples illustrate the following principle:

20. Principle IV. *To multiply the sum or difference of two numbers by a given number, multiply each of the numbers separately by the given number, and add or subtract the products.*

21. Principles III and IV should be carefully contrasted, as in the following example:

$$2(2 \cdot 3 \cdot 5) = 4 \cdot 3 \cdot 5 = 2 \cdot 6 \cdot 5 = 2 \cdot 3 \cdot 10,$$

but

$$2(2 + 3 + 5) = 4 + 6 + 10.$$

In multiplying the product of several numbers we operate upon *any one of them*, but in multiplying the sum or difference of numbers we operate upon *each of them.*

EXERCISES AND PROBLEMS

1. Multiply $5 + 7 + 11$ by 3 without first adding, and then check by performing the addition before multiplying.

2. Multiply $m + n$ by 4 and check for $m = 5$, $n = 7$.

$$4(m + n) = 4m + 4n$$

Check. $4(5 + 7) = 4 \cdot 12 = 48$, also

$$4 \cdot 5 + 4 \cdot 7 = 20 + 28 = 48.$$

3. Multiply $r + s + x$ by a and check for $r = s = x = a = 2$.

4. Multiply $a + b + c$ by m and check for $a = 1$, $b = 3$, $c = 5$, $m = 4$.

5. Multiply $x + y$ by r and check for $x = 2$, $y = 4$, $r = 6$.

Where letters are involved, check the results in the following by substituting convenient values:

6. $8(13-5)$.
7. $71(12+41-36)$.
8. $9(a-b+8)$.
9. $3(a+x+y-17)$.
10. $m(a+b-c)$.
11. $a(18-7)$.
12. $2(13+8+9-21)$.
13. $7(3+8+9-a)$.
14. $5(17+a-b)$.
15. $32(x-y+z)$.
16. $8(2a-3b+4c)$.
17. $37(3x-2y-z)$.
18. $13(5r-3x+t)$.
19. $20(2x+3x-5y)$.
20. $7(11s-2s+3t)$.
21. $35(x-2y+3z)$.
22. $78(10m+11n+2r)$.
23. $4(25x+32x-y)$.
24. $3(13x+14y-t)$.
25. $7(4a-3b+c)$.

26. Find two consecutive integers such that 3 times the first plus 7 times the second equals 217.

27. Find two consecutive integers such that 7 times the first plus 4 times the second equals 664.

28. There are three numbers such that the second is 17 less than the first, and the third is 8 times the second. The sum of the first and third is 89. What are the numbers?

29. The number of representatives and senators together in the United States Congress is 476. The number of representatives is 26 more than 4 times the number of senators. Find the number of each.

30. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

31. The sum of the horse-powers of the steamships *Campania* and *Mauritania* is 102 thousand. The *Mauritania* has 12 thousand horse-power more than twice that of the *Campania*. What is the horse-power of each ship?

32. The population of Wyoming (census of 1900) was 50 thousand more than that of Nevada, and the population of Utah was 93 thousand more than twice that of Wyoming. The population of Utah was 277 thousand. Find the population of Nevada and of Wyoming.

33. It is 73 miles farther from Newark to Philadelphia than from New York to Newark; and it is one mile more than 10 times as far from Philadelphia to Chicago as from Newark to Philadelphia. The sum of the distances from New York to Newark and from Philadelphia to Chicago is 830 miles. Find each of the three distances, and the total distance from New York to Chicago.

Let $d =$ number of miles from New York to Newark.
Then $d + 73 =$ number of miles from Newark to Philadelphia,
and $10(d + 73) + 1 =$ number of miles from Philadelphia to Chicago.
Hence $d + 10(d + 73) + 1 = 830.$

34. In the championship season of 1906 the Chicago National League baseball team lost 20 games less than New York, and Pittsburg lost 12 less than twice as many games as Chicago. Pittsburg and New York together lost 116 games. How many did each of the three teams lose?

35. Pikes Peak is 3282 feet higher than Mt. Ætna, and Mt. Everest is 708 feet more than twice as high as Pikes Peak. The sum of the altitudes of Mt. Ætna and Mt. Everest is 39,867 feet. Find the altitude of each of the three mountains.

36. The altitude of Chimborazo is 22,820 feet less than 3 times that of Mt. Shasta. What is the altitude of each mountain if Chimborazo is 6060 feet higher than Mt. Shasta?

Let $x =$ the number of feet in the altitude of Mt. Shasta. Then $3x - 22820 =$ altitude of Chimborazo, and $3x - 22820 - x = 6060$.

37. The distance of Mars from the sun is 39 million miles less than 5 times as great as that of Mercury from the sun. Mars is 105 million miles farther from the sun than Mercury. What is the distance of each planet from the sun?

38. The population of New York City (estimate of Census Bureau, 1904) was 5 thousand more than 11 times that of Pittsburg. If 21 times the population of Pittsburg is subtracted from twice that of New York, the remainder is 363 thousand. Find the population of each city.

39. The standing army of France (1906) was 383 thousand less than twice that of Germany. If 7 times the number of men in the German army is subtracted from 6 times the number of men in the French army, the remainder is 177 thousand. Find the number of men in each army.

DIVISION OF A PRODUCT

22. The division of the product of several factors by a given number may be performed in various ways:

E.g. $(4 \cdot 6 \cdot 10) \div 2 = 240 \div 2 = 120.$

Also $(4 \cdot 6 \cdot 10) \div 2 = 2 \cdot 6 \cdot 10 = 120,$

$(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 3 \cdot 10 = 120,$

and $(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 6 \cdot 5 = 120.$

In each case only one factor is divided, and since any factor may be selected, we naturally choose one which exactly contains the divisor if possible.

Perform each of the following divisions in more than one way where possible:

1. $(5 \cdot 8 \cdot 3) \div 2.$
2. $20\,abc \div 4.$
3. $12\,abc \div 3.$
4. $(11 \cdot 20 \cdot 16) \div 4.$
5. $14\,xyz \div 7.$
6. $12\,abc \div c.$
7. $(10 \cdot 35 \cdot 3) \div 5.$
8. $14\,xyz \div x.$
9. $(12 \cdot 40 \cdot 13) \div 8.$

These examples illustrate the following principle:

23. **Principle V.** *To divide the product of several factors by a given number divide any one of the factors by that number, leaving the other factors unchanged.*

Principle V is already known in arithmetic in the process called cancellation.

Thus, in the fraction $\frac{2 \cdot 6 \cdot 9}{3}$, 3 may be canceled out of either 6 or 9, giving $\frac{2 \cdot 6 \cdot 9}{3} = 2 \cdot 2 \cdot 9$ or $2 \cdot 6 \cdot 3$.

Principle V is necessary in the solution of problems like the following:

Illustrative Problem. There are three numbers whose sum is 26. The second is 8 times the first, and the third is one-half the second. Find each of the numbers.

Solution. Let x represent the first number,

then $8x$ represents the second number,

and $\frac{8x}{2}$ represents the third number.

Hence, $x + 8x + \frac{8x}{2} = 26$, the sum of the numbers.

By Principle V, $x + 8x + 4x = 26$, since $8x \div 2 = 4x$.

By Principle I, $13x = 26$.

Hence, $x = 2$, the first number,

$8x = 16$, the second number,

and $\frac{8x}{2} = \frac{8 \cdot 2}{2} = 8$, the third number.

EXERCISES AND PROBLEMS

1. Multiply $2x + 3y$ by 5. Check for $x = 7$, $y = 11$.

2. Multiply $3a + y$ by 8. Check for $a = 2$, $y = 3$.

3. Divide $3 \cdot 7 \cdot 18$ by 6 by means of Principle V.

4. Multiply $7 \cdot 56 \cdot 5$ by 6 by means of Principle III.

5. Divide $21 \cdot 36 \cdot 42$ by 7, leaving the result in two different forms.

6. Multiply $5 \cdot 13 \cdot 27$ by 3, leaving the result in three different forms.

7. Divide $7a \cdot 14b \cdot 21c$ by 7 in three different ways.

8. Add $5a$, $\frac{12a}{2}$, $\frac{21a}{3}$, and $\frac{18a}{6}$, using Principles V and I.

9. From $\frac{28xy}{4}$ subtract $\frac{21xy}{7}$, using Principles V and II.

10. From $\frac{14a}{2} + \frac{10a}{5}$ subtract $\frac{6a}{3}$.

11. Find the sum of $\frac{16x}{8}$, $\frac{20x}{5}$, $\frac{16x}{4}$, $7x$, and $3x$.

12. Find the sum of $\frac{100rs}{10}$, $\frac{90rs}{9}$, and $\frac{25rs}{5}$.

13. From $25xy$ subtract $\frac{13xyz}{z}$.

14. Add $\frac{18abc}{a}$, $20bc$, and $\frac{30bc}{10}$.

15. Add $\frac{17rst}{s}$, $\frac{18art}{a}$, and $\frac{20rtu}{u}$.

16. From $\frac{150xy}{y} + 17x$ subtract $\frac{79xz}{z}$.

17. From $179m + \frac{39mn}{n}$ subtract $\frac{25am}{a}$.

18. From $\frac{24(a+b)}{6} + \frac{14(a+b)}{2}$ subtract $7(a+b)$.

19. Cleveland had (estimate of Census Bureau, 1904) 8 times as many inhabitants as Portland, Maine. If twice the population of Portland is added to $\frac{1}{4}$ that of Cleveland, the sum is 212 thousand. Find the population of each city.

20. One cubic foot of a certain kind of brick weighs as much as 3 cubic feet of cedar. The combined weight of $\frac{1}{6}$ of a cubic foot of brick and 5 cubic feet of cedar is 187 pounds. Find the weight per cubic foot of each.

21. It is 8 times as far from Philadelphia to Louisville as from Philadelphia to Baltimore. If $\frac{1}{4}$ the distance from Philadelphia to Louisville is added to 3 times that from Philadelphia to Baltimore, the sum is 485 miles. Find each of the two distances.

22. Coinage silver weighs 4 times as much per cubic inch as feldspar. The combined weight of $\frac{1}{2}$ a cubic inch of silver and 7 cubic inches of feldspar is .846 pound. What is the weight of a cubic inch of each?

23. It is 3 times as far from New York to Washington, D.C., as from New York to New Haven, and it is 14 times as far from New York to Seattle as from New York to Washington. If $\frac{1}{7}$ the distance from New York to Seattle is added to 5 times that from New York to New Haven, the sum is 836 miles. Find the distance from New York to each of the other cities.

24. The population of Washington, D.C. (estimate of Census Bureau, 1904), was 9 times that of Galveston, Texas, and the population of Savannah, Georgia, was 33 thousand less than $\frac{1}{3}$ that of Washington. The combined population of Galveston and Savannah was 99 thousand. Find the population of each city.

25. A cubic foot of steel weighs 17 times as much as a cubic foot of yellow pine. The combined weight of 11 cubic feet of pine and 3 cubic feet of steel is 1773.2 pounds. Find the weight of 1 cubic foot of each.

DIVISION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

24. In dividing the sum or difference of two numbers by a given number, when these are represented by Arabic figures, the process may be carried out in two ways. Thus,

(1) $(12 + 8) \div 2 = 20 \div 2 = 10,$

or $(12 + 8) \div 2 = 12 \div 2 + 8 \div 2 = 6 + 4 = 10.$

(2) $(20 - 12) \div 4 = 8 \div 4 = 2,$

or $(20 - 12) \div 4 = 20 \div 4 - 12 \div 4 = 5 - 3 = 2.$

The same result is obtained in each case whether the numbers in the parenthesis are first added (or subtracted) and the result then divided by the given number; or the numbers in parenthesis are first divided separately and then the quotients added (or subtracted).

If the numbers in the dividend are represented by letters, the division can usually be carried out only in the second manner shown above.

E.g. $(r + t) \div 5 = r \div 5 + t \div 5$, or, $\frac{r+t}{5} = \frac{r}{5} + \frac{t}{5}$.

This is read: the sum of r and t divided by 5 equals r divided by 5 plus t divided by 5.

In this manner perform each of the following divisions in two ways when possible:

1. $(16 + 12) \div 4$.
2. $(20a - 10b) \div 5$.
3. $(36x - 24y) \div 6$.
4. $(108y - 72z) \div 12$.
5. $(50x + 75x) \div 25$.
6. $(x + y + z) \div 3$.
7. $(15t - 3t + 18t) \div 3$.
8. $(18x + 36y - 21t) \div 3$.
9. $(m - n - r) \div a$.
10. $(m + n + r) \div b$.

These examples illustrate the following principle:

25. Principle VI. *To divide the sum or difference of two numbers by a given number divide each number separately and find the sum or difference of the quotients.*

Principle VI is necessary in problems such as the following:

Illustrative Problem. The population of Iceland is 40,500 less than 10 times that of Greenland. The population of Greenland plus $\frac{1}{5}$ that of Iceland is 27,600. Find the population of each.

Solution. Let x be the number of inhabitants in Greenland.

Then $10x - 40500$ is the number in Iceland.

Hence $$x + \frac{10x - 40500}{5} = 27600.$$

By Principle VI, $$x + 2x - 8100 = 27600.$$

By Principle I, $$3x - 8100 = 27600.$$

Adding 8100 to each member and observing that $8100 - 8100 = 0$,

$$3x = 35700.$$

Therefore $x = 11900$, the population of Greenland,

and $10x - 40500 = 78500$, the population of Iceland.

Check. $$11900 + \frac{78500}{5} = 27600.$$

26. Principles V and VI should be carefully contrasted:

Thus: $$\frac{12 \cdot 18 \cdot 24}{3} = 4 \cdot 18 \cdot 24 = 12 \cdot 6 \cdot 24 = 12 \cdot 18 \cdot 8,$$

while $$\frac{12 + 18 + 24}{3} = 4 + 6 + 8.$$

That is, in dividing the product of several numbers we operate upon *any one of them* as found convenient, but in dividing the sum of several numbers we must operate upon *each of them.*

EXERCISES AND PROBLEMS

1. Divide $72 + 56$ by 8 without first adding.

2. Divide $144 - 36$ by 12 without first subtracting.

3. Divide $r + t$ by 5 and check the quotient when $r = 15$, $t = 25$; also when $r = 60$, $t = 75$.

4. Multiply $7 + 9$ by 3 without first adding 7 and 9.

5. Multiply $25 - 8$ by 5 without first subtracting.

6. Find the product of 12 and $a + b$, checking the result when $a = 5$, $b = 7$.

Perform the following indicated operations:

7. $3(a+b+c+d)$. Check for $a=1$, $b=2$, $c=3$, $d=4$.

8. $7(r-s+t-x)$. Check for $r=t=5$, $s=x=4$.

9. $(m+n+r)\div 4$. Check for $m=64$, $n=32$, $r=8$.

10. $(x+y+z)\div 5$. Check for $x=100$, $y=50$, and $z=25$.

11. Add $5(a+b+c)$ and $\dfrac{6a+12b+24c}{3}$.

12. From $25(x+y+z)$ subtract $\dfrac{100x+50y+25z}{25}$.

13. Add $\dfrac{6ab+7ac+ad}{a}$ and $\dfrac{12b+15c+9d}{3}$.

14. Perform the divisions: $\dfrac{33x-44y}{11}$ and $\dfrac{78t-39s}{13}$.

15. Divide $7mxy+3nxy-2mny$ by y, and check for $m=2$, $n=3$, $x=4$, $y=5$.

16. Add $\dfrac{21ax+49by+56ab}{7}$ and $\dfrac{24ax+56by+64ab}{8}$, and check for $a=b=c=2$, $x=y=3$.

17. If the sum of 4 times a number and 32 be divided by 2 the result is 30. Find the number.

18. The melting temperature of glass is 548 degrees (Centigrade) lower than 4 times that of zinc. One-half the number of degrees at which glass melts plus 7 times the number at which zinc melts equals 3434. Find the melting point of each.

19. The melting temperature of silver is 496 degrees (Centigrade) lower than that of nickel. Five times the number of degrees at which nickel melts plus 7 times the number at which silver melts equals 13,928. Find the melting point of each metal.

20. The population of Paris (1904) was 1360 thousand less than twice that of Berlin. The sum of their populations was 4730 thousand. Find the population of each city.

21. A cubic foot of nickel weighs 1288 pounds less than 4 cubic feet of tin. One-half a cubic foot of nickel plus 1 cubic foot of tin weighs 724 pounds. Find the weight per cubic foot of each metal.

22. A cubic foot of gold weighs 2730 pounds less than 6 cubic feet of silver. One-third of a cubic foot of gold together with 5 cubic feet of silver weighs 3675 pounds. Find the weight per cubic foot of each metal.

23. The population of Japan (1904) was 106 million less than 12 times that of Manchuria. If the population of Manchuria be subtracted from $\frac{1}{2}$ that of Japan, the remainder is 12 million. Find the population of each country.

24. The population of Panama (1900) was 1160 thousand less than 3 times that of Nicaragua (1905). Three times the population of Panama plus twice that of Nicaragua is 2020 thousand. Find the population of each country.

25. The population of the Philippines (1903) was 400 thousand less than 50 times that of Hawaii. One twenty-fifth of the population of the Philippines plus the population of Hawaii is 464 thousand. What was the population of each?

26. One gallon of benzine weighs 45.4 pounds less than 8 gallons of alcohol. The weight of $\frac{1}{2}$ a gallon of benzine and 2 gallons of alcohol is 16.9 pounds. Find the weight of one gallon of each liquid.

27. During the season 1906 the American League baseball team of Chicago won 201 games less than 6 times as many as Boston. One-third the number of games won by Chicago plus 7 times the number of games won by Boston equals 374. How many games did each team win?

28. The altitude of Aconcagua is 70,800 feet less than 6 times that of Mt. Blanc. One-sixth the altitude of Aconcagua added to that of Mt. Blanc equals 19,760 feet. Find the altitude of each mountain.

29. The area of Great Britain is 30,387 square miles less than 12 times that of the Netherlands, and the area of Japan is 42,065 square miles less than 15 times that of the Netherlands. One-third the area of Great Britain plus $\frac{1}{5}$ the area of Japan is 69,994 square miles. Find the area of each country.

30. The diameter of the earth is 1918 miles more than twice that of Mercury, and the diameter of Venus is 1700 miles more than twice that of Mercury. The diameter of the earth plus $\frac{1}{2}$ that of Venus equals 11,768 miles. Find the diameter of each planet.

31. The diameter of Jupiter is 500 miles more than 20 times that of Mars, and the diameter of Saturn is 4200 miles more than 16 times that of Mars. One-tenth the diameter of Jupiter plus $\frac{1}{2}$ that of Saturn is 45,150 miles. Find the diameter of each planet.

32. The diameter of Neptune is 29,000 miles less than twice that of Uranus. One-half the diameter of Neptune plus 4 times that of Uranus is 145,000 miles. Find the diameter of each planet.

From the last three problems make a table of the diameters of all the planets.

NUMBER EXPRESSIONS IN PARENTHESES

27. When a number expression is inclosed in a parenthesis, it is sometimes possible to perform the operations indicated within the parenthesis and thus to remove it.

E.g. $6 + (7 + 10 - 9) = 6 + 8 = 14.$

When, however, the operations within the parenthesis cannot be carried out, the parenthesis may sometimes be removed by Principle IV or VI.

E.g. $5(2n+7r)=10n+35r$, and $(20x-16y)\div 4=5x-4y$.

In these examples the operations upon the parentheses are multiplication or division. The cases in which the operations on the parentheses are addition or subtraction are considered in the following examples:

(1) $5+(3+4)=5+7=12,$

also $5+(3+4)=5+3+4=8+4=12.$

(2) $10+(7-3)=10+4=14,$

also $10+(7-3)=10+7-3=17-3=14.$

(3) $20-(7+2)=20-9=11,$

also $20-(7+2)=20-7-2=13-2=11.$

(4) $20-(7-2)=20-5=15.$

also $20-(7-2)=20-7+2=13+2=15.$

From (1) it appears that the expression, $3+4$, may be added to 5 by first adding 3 and then adding 4.

From (2) it is seen that the expression, $7-3$, may be added to 10 by first adding 7 and then subtracting 3.

From (3) it is seen that the expression, $7+2$, may be subtracted from 20 by first subtracting 7 and then subtracting 2.

From (4) it is evident that the expression, $7-2$, may be subtracted from 20 by first *subtracting* 7 and then *adding* 2, since 7 is 2 more than the number which was to be subtracted.

In like manner perform each of the following operations in two ways:

1. $18+(5+2+3)$.
2. $28+(7-3+4)$.
3. $32-(5+3)$.
4. $40-(6+7+3)$.
5. $55-(16-7)$.
6. $101+(73-22)$.
7. $25n+(22n-3n)$.
8. $18x-(6x-2x)$.
9. $16r-(7r+2r)$.

These examples illustrate the following principle:

28. Principle VII. *A number expression consisting of two or more numbers connected by the signs + or − may be added to another given number by adding each number with the plus sign and subtracting each number with the minus sign. Such a number expression may be subtracted from a given number by subtracting each number with the plus sign and adding each number with the minus sign.*

Principles IV, VI, and VII are useful in removing parentheses from number expressions when the values of the numbers involved are not known.

Instead of a parenthesis, a bracket [ ], or a brace { }, may be used. Thus, $2(6+4) = 2[6+4] = 2\{6+4\}$.

In expressions involving parentheses the operations within the parentheses should be performed first if possible. Then perform the indicated multiplications and divisions, and finally the remaining additions and subtractions.

E.g. Given $7 + 15 \div (9 - 4) - 4[7 + 11] \div (29 - 20)$.

Performing operations within the parentheses, this reduces to:

$$7 + 15 \div 5 - 4 \cdot 18 \div 9.$$

Performing multiplications and divisions, we have $7 + 3 - 8$.

Performing the remaining additions and subtractions, the given expression reduces to 2.

EXERCISES

In performing the following indicated operations, state in each case what principles are used.

1. $8(7+4+8+2)$.
2. $6(7-3) \div 2$.
3. $3(2a-4) - 4(a-6)$.
4. $14 + 4(8+4) \div (32-20)$.

The minus sign preceding $4(a-6)$ indicates that this whole product is to be subtracted. Hence, using Principle IV, we have $6a - 12 - (4a - 24)$. Then, using Principle VII, this becomes $6a - 12 - 4a + 24 = 2a + 12$.

5. $8-3(4-2)+6+3[6+1]$.

6. $16 \div [3+1] - \{12-3\} \div 3$.

7. $5(17-5)+18 \div (8-2)-(21+14) \div 7$.

8. $7(a+b)+6(a+b)$.

9. $5(a+b)+3(a+b+c)$.

10. $16(r+t)+11(r+t)$.

11. $15s-3(r+s)$.

12. $15(r+s)-3(r-s)$.

13. $20-(x+y)$.

14. $50x-25(x-y)$.

15. $12y-6(x-2y)$.

16. $11t+5(2t-1)-3(2+2)$. Check for $t=2$.

17. $(12t-6u) \div 2+3t-2u$. Check for $t=1$, $u=1$.

18. $3(5x-7y)+(21x-28y) \div 7$. Check for $x=2, y=1$.

19. $8(r-s)+5(2r+s)-3(r+s)$. Check for $r=1, s=1$.

20. $10(x+y) \div 5+6(x-y) \div 3$. Check for $x=2$, $y=1$.

21. $5(h+k)+2(h+k)+3(h+k)$.

Use Principle I, then IV; also IV, then I.

22. $5(7x-4y)+9(9x-5y)$.

23. $8(r-s)-2(r-s)$.

24. $9(3p-q)-8(3p-q)$.

Use Principle II, then IV; also reverse this order.

25. $(16x-12x) \div 4$.

26. $(18ab-12abx) \div a$.

Use first Principle VI, then II; also II, and then V.

27. $9(6+4)-3(6-4)$.

28. $(5axy-3cx) \div x$.

29. $9(6abc+4xyz)-3(6abc-4xyz)$.

30. $8(5x-y+2z)-11(3x+2y-7z)$.

31. $3[3(a+b+c)-2(a-b+c)]$.

First remove the parentheses, then the bracket.

32. $7\{8x-(2y-3x)+(2x-4y)\}$.

PROBLEMS

1. There are three numbers whose sum is 80. The second is 3 times the first, and the third twice the second. What are the numbers?

2. A man has three buildings whose total value is $46,800. The second building cost $800 less than the first, and the third cost twice as much as the second. What is the cost of each building?

3. The population of Connecticut (Census of 1900) was 50 thousand more than twice that of Rhode Island, and the population of Massachusetts was 81 thousand more than 3 times that of Connecticut. The population of Massachusetts minus that of Connecticut was 1897 thousand. Find the population of each state.

4. The population of Indiana (Census of 1900) exceeded that of Iowa by 284 thousand, and the population of Illinois was 210 thousand less than twice that of Indiana. The population of Illinois minus that of Indiana was 2306 thousand. Find the population of each state.

5. The melting point of copper is 250 degrees (Centigrade) lower than 4 times that of lead. Ten times the number of degrees at which lead melts minus twice the number at which copper melts equals 1152. What is the melting point of each metal?

6. The melting point of iron is 450 degrees higher than 5 times that of tin. Three times the number of degrees at which iron melts plus 7 times the number at which tin melts equals 6410. Find the melting point of each metal.

7. In 1904 the gold product of Africa was 11 million dollars more than 3 times that of Russia, and the gold product of Australia was 84 million less than twice that of Africa. Russia and Australia together produced 113 million. How much did each country produce?

8. In 1904 the value of the silver produced in the United States was 5 million dollars more than 14 times as much as that of Canada, and the product of Mexico was 1 million less than 16 times that of Canada. Twice the product of the United States minus that of Mexico equals 71 million. How much did each country produce?

9. The Nile is 100 miles more than twice as long as the Danube. Ten times the length of the Danube minus 4 times the length of the Nile equals 3400 miles. How long is each river?

10. The number of first-class battleships in the United States navy (1906) was 22 less than 5 times the number of protected cruisers. Twelve times the number of cruisers less twice the number of battleships equals 60. Find the number of each.

11. The number of torpedo boats in the United States navy (1906) was 77 less than 7 times as great as the number of torpedo boat destroyers. Three times the number of torpedo boats minus 4 times the number of destroyers equals 41. Find the number of each.

12. The weight of a cubic foot of spruce is 16 pounds more than that of a cubic foot of cork, and the weight of a cubic foot of dry live oak is 5 pounds more than twice that of spruce. One cubic foot of oak weighs 36 pounds more than one cubic foot of spruce. Find the weight of a cubic foot of each.

13. In 1904 Canada produced 3 million dollars more of gold than Mexico, and the United States produced 17 million more than 4 times as much as Canada. The combined product of Mexico and the United States was 94 million. How much did each country produce?

14. The value of the copper produced in the United States in 1904 was 5 million dollars more than the value of the crude petroleum, and the value of the bituminous coal was 94 million more than twice the value of the copper. Nine times the value of the copper minus twice the value of the coal equals 342 million. Find the value of each.

15. The pressure developed in the chamber of a modern twelve-inch gun is 21 thousand pounds per square inch more than that developed in an ordinary hunting rifle. The maximum pressure which gunpowder *can* develop is 18 thousand pounds per square inch less than 3 times as great as that developed in the twelve-inch gun. The sum of the three pressures is 151 thousand pounds. Find each pressure.

16. The standing army of Australia (1906) was 14 thousand more than that of Canada, and the standing army of Great Britain was 47 thousand more than 4 times that of Australia. Great Britain's army contained 287 thousand men. How many men were there in the armies of Canada and Australia?

IDENTITIES AND EQUATIONS

29. In the preceding pages equations have been freely used, and certain simple methods have been employed in solving problems by means of them. We now proceed to a more detailed study of these methods and of the equation itself.

30. Equalities in which letters are used as number symbols are of two kinds as shown by the following examples:

(1) $3n + 4n = 7n$ is a true statement no matter what number is represented by n. $3(5x + 6y) = 15x + 18y$ holds for all values which may be assigned to x and y.

(2) $w + 2(w+5) = 58$ is a true statement if, and only if, $w = 16$. If w is replaced by any number less than 16, the number expression on the left is less than 58, and if w is replaced by any number greater than 16, the result is greater than 58.

31. The equality $w + 2(w + 5) = 58$ is said to be **satisfied** by $w = 16$, because this value of w reduces both members to the same number, 58.

32. Definition. An equality which is satisfied, no matter what numbers are substituted for one or more of its letters, is called an **identity** with respect to those letters.

E.g. $3n + 4n = 7n$ is an identity with respect to n.
$a(b + c) = ab + ac$ is an identity with respect to a, b, and c.

When it is especially desired to distinguish an equality as an identity, the equality sign is written $\equiv$.

Each of the Principles I to VII may be thus stated in symbols as identities. Thus,

$$\text{I, } 4n + 6n \equiv 10n; \qquad \text{II, } 12n - 5n \equiv 7n;$$
$$\text{III, } 5(4ab) \equiv 20ab; \qquad \text{IV, } 5(a \pm b) \equiv 5a \pm 5b;$$
$$\text{V, } 30ab \div 6 \equiv 5ab; \qquad \text{VI, } (16a \pm 20b) \div 4 \equiv 4a \pm 5b;$$
$$\text{VII, } x + (a - b) - (c - d) \equiv x + a - b - c + d.$$

33. Definition. An equality which is satisfied only when certain particular values are given to one or more of its letters is called a **conditional equality** with respect to those letters.

E.g. $3x + 5 = 35$ is an equality only on the condition that $x = 10$.
$x + y = 10$ is an equality for certain pairs of values like 1 and 9, 2 and 8, 3 and 7, 5 and 5, but certainly not for all values of x and y; for instance, not for 3 and 8.

34. A conditional equality is called an **equation**, and a letter whose particular value is sought to satisfy an equation is called an **unknown number** in that equation, or simply an **unknown.**

The equations at present considered contain only one unknown.

35. To solve an equation in one unknown is to find the value or values of the unknown which satisfy it.

The following example exhibits the process of solving an equation which contains one unknown and which is satisfied by one value only of the unknown:

Given	$w + 2(w + 5) = 58.$	(1)
By Principle IV,	$w + 2w + 10 = 58.$	(2)
By Principle I,	$3w + 10 = 58.$	(3)
Subtracting 10 from both members,	$3w = 48.$	(4)
Dividing both sides by 3,	$w = 16.$	(5)

Check. Putting $w = 16$ in (1), $16 + 2(16 + 5) = 16 + 2 \cdot 21 = 58.$

Explanation. Any value of w which satisfies (1) also satisfies (2) and (3), since the expressions in the left members of (1), (2), and (3) represent the same number expressed in different forms. Any value of w which satisfies (3) also satisfies (4), for if 10 more than $3w$ is 58, then $3w$ must be 48. Any value of w which satisfies (4) also satisfies (5), for if $3w$ is 48, then w is $\frac{1}{3}$ of 48.

By similar considerations the value of w which satisfies (5) may be shown to satisfy (4), (3), (2), and (1). Hence $w = 16$ is the solution of the given equation.

The above solution illustrates the following principle:

36. Principle VIII. *An equation may be changed into another equation such that any value of the unknown which satisfies one also satisfies the other, by means of any of the following operations:*

(1) *Adding the same number to both members.*

(2) *Subtracting the same number from both members.*

(3) *Multiplying both members by the same number.*

(4) *Dividing both members by the same number.*

(5) *Changing the form of either member in any way which leaves its value unaltered.*

The operations under Principle VIII are hereafter referred to in detail by means of the initial letters, A for addition, S for subtraction, M for multiplication, D for division, and F for form changes.

NOTE. — It is not permissible to multiply or divide the members of an equation by any expression which is equal to zero. See Advanced Course.

37. The operations involved in Principles I to VII are all **form changes** which leave the value of the expression unaltered.

E.g. $4(m+n)$ by Principle IV has the same value as $4m+4n$.

There are other form changes which are already familiar in arithmetic.

E.g. $2+4=4+2$, or in general $a+b=b+a$. Likewise $2\cdot 7=7\cdot 2$, or in general $ab=ba$. That is, the order in which numbers are added or multiplied is immaterial.

Again, $3+4+6=(3+4)+6=3+(4+6)$, or in general $a+b+c=(a+b)+c=a+(b+c)$, and likewise $a\cdot b\cdot c=(a\cdot b)\cdot c=a\cdot(b\cdot c)$; *i.e.* numbers to be added or multiplied may be grouped in any manner desired. Still other form changes will be learned later.

38. Principle VIII is further illustrated as follows:

On the scale pans of a common balance are placed objects of uniform weight, say tenpenny nails. The scales balance only when the weights are the same in both pans; that is, when the number of nails is the same.

If now the scales are in balance, they will remain so under two kinds of changes in the weights:

(*a*) When the number of nails in the two pans is increased or diminished by the same number; corresponding to the operations *A, S, M, D,* on the members of an equation.

(*b*) When the number in each pan is left unaltered but the nails are rearranged in groups or piles in any manner; corresponding to the operations *F* on the members of an equation.

The equation, then, is like a balance, and its members are to be operated upon only in such ways as to preserve the balance.

DIRECTIONS FOR WRITTEN WORK

39. The solution of an equation consists in deducing, by means of Principles I to VIII, another equation whose first member contains the unknown only and whose second member does not contain the unknown. The successive steps should be written down as in the following example:

Solve $6(4n - 3) + 25(n + 1) = 50 + 31n + 2(3 - n) - 9$. (1)

By F, using III and IV, we obtain from (1)

$$24n - 18 + 25n + 25 = 50 + 31n + 6 - 2n - 9. \quad (2)$$

By F, I and II, we obtain from (2)

$$49n + 7 = 29n + 47. \quad (3)$$

Subtracting 7 and $29n$ from each member of (3) and using Principle II, we have

$$20n = 40. \quad (4)$$

Dividing each member of (4) by 20,

$$n = 2. \quad (5)$$

Check. Substitute $n = 2$ in equation (1).

For convenience this work can be abbreviated as follows:

	$6(4n - 3) + 25(n + 1) = 50 + 31n + 2(3 - n) - 9.$	(1)
By F, III, IV,	$24n - 18 + 25n + 25 = 50 + 31n + 6 - 2n - 9.$	(2)
By F, I, II,	$49n + 7 = 29n + 47.$	(3)
By $S \mid 7, 29n$,	$20n = 40.$	(4)
By $D \mid 20$,	$n = 2.$	(5)

$S \mid 7, 29n$ means that 7 and $29n$ are each to be subtracted from both members of the preceding equation. $D \mid 20$ means that the members of the preceding equation are to be divided by 20.

Similarly, in case we wish to indicate that 6 is to be added to each member of an equation, we would write $A \mid 6$, and if each member is to be multiplied by 8, we would write $M \mid 8$. It is important that the nature of each step be recorded in some such manner.

(2) Solve $17n + 4(2 + n) - 6 = 5(4 + n) - 5 + 3n.$ (1)

By F, IV,	$17n + 8 + 4n - 6 = 20 + 5n - 5 + 3n.$	(2)
By F, I,	$21n + 2 = 15 + 8n.$	(3)
By $S \mid 2, 8n$,	$21n - 8n = 15 - 2.$	(4)
By F, II,	$13n = 13.$	(5)
By $D \mid 13$,	$n = 1.$	(6)

Check. Substitute $n = 1$ in equation (1);

then $$17 + 12 - 6 = 25 - 5 + 3,$$

or $$23 = 23.$$

An equation may be translated into a problem. For example, the equation $21x + 2 = 15 + 8x$ may be interpreted as follows: Find a number such that 21 times the number plus 2 is 15 greater than 8 times the number.

EXERCISES AND PROBLEMS

Solve the following equations, putting the work in a form similar to the above and checking each result. Translate the first twenty into problems.

1. $13x - 40 - x = 8.$
2. $3x + 9 + 2x + 6 = 18 + 4x.$
3. $5x + 3 - x = x + 18.$
4. $13y + 12 + 5y = 32 + 8y.$
5. $4m + 6m + 4 = 9m + 6.$
6. $7m - 18 + 3m = 12 + 2m + 2.$
7. $3y - 4 + 2y - 6 = y + 7 + y + 3 + 10.$
8. $5x + 3 + 2x + 3 = 2x + 5 + 3x + 3 + x.$
9. $2x + 4x + 9 + x + 6 = 20 + 3x + 5 + 2x.$
10. $18 + 6m + 30 + 6m = 4m + 8 + 12 + 3m + 3 + m + 29.$

11. $y + 72 + 45\,y = 106 + 12\,y$.

12. $42\,x - 56 = 20\,x + 10$.

13. $6\,x + 8 + 3\,x + 4 + 5\,x = 7\,x + 32 + x - 20$.

14. $32\,x - 4 + 7\,x = 58 + 3\,x + 5\,x$.

15. $12\,m - 3 - 3\,m = 32 + 2\,m$.

16. $15\,m + 3 - 2\,m + 7 = 3\,m + 60$.

17. $a + 7 + 3\,a = 2\,a + 45$.

18. $5\,b - 30 + 6\,b = 3\,b + 90$.

19. $3\,c + 18 + 14\,c = 6\,c + 51$.

20. $17\,x + 4 + 3\,x = 7\,x + 30$.

21. $7(m + 6) + 10\,m = 42 - 8(2\,m + 2) + 181$.

22. $20 - 3(x - 4) + 2\,x = 2\,x + 17$.

23. $(8\,x + 4) \div 2 + 7\,x = 4\,x + 9$.

24. $6\,x + 4(4\,x + 2) = 85 - 3(2\,x + 7)$.

25. $8 + 7(6 + 6\,n) + 2\,n = 2(4\,n + 5) + 18\,n + 49$.

26. $5(9\,x + 3) + 6\,x = 24\,x - 4(3\,x + 2) + 36$.

27. $\dfrac{7(12\,x + 8)}{4} + 13 + 5\,x - 6 = 47$. Apply V and VI.

28. $\dfrac{12(5 + 4\,x)}{6} - \dfrac{5(6 + 4\,x)}{2} + 50 = x + 18$.

29. $15 + \dfrac{21\,(3 + x)}{7} + \dfrac{2(6 + 18\,x)}{3} = \dfrac{3(9\,x + 12)}{3} + 28$.

30. $\dfrac{11(5\,x + 25)}{5} + \dfrac{3(6\,x - 2)}{2} = \dfrac{7(4\,x + 8)}{4} + \dfrac{12\,x + 36}{3} + 35$.

EQUATIONS INVOLVING FRACTIONS

40. Solve $n + \frac{n}{2} + \frac{n}{3} = 88.$ (1)

First Solution. The coefficients of n are 1, $\frac{1}{2}$, and $\frac{1}{3}$.

Applying Principle I, $1\frac{5}{6}\, n = 88.$ (2)

By $D \mid 1\frac{5}{6}$, $n = 88 \div 1\frac{5}{6} = 48.$ (3)

Second Solution. Multiply both members of (1) by 6.

That is, by $M \mid 6$, $6\,n + \frac{6\,n}{2} + \frac{6\,n}{3} = 528.$ (2)

By F, V, $6\,n + 3\,n + 2\,n = 528.$ (3)

By F, I, $11\,n = 528.$ (4)

By $D \mid 11$, $n = 48$, as before. (5)

The object is to multiply both members of the equation by such a number as will cancel each denominator. Hence the multiplier must contain each denominator as a factor.

Evidently 12 or 18 might have been chosen for this purpose, but not 8 or 10. 6 is the smallest number which will cancel both 2 and 3, and hence this was chosen as the multiplier.

41. The process explained in the second solution above is called **clearing of fractions.**

As another illustration solve the equation

$$\frac{3\,x}{4} + \frac{x}{2} + \frac{5\,x}{9} = 65. \qquad (1)$$

Here the smallest multiplier available is 36.

Hence by $M \mid 36$, $\frac{36 \cdot 3\,x}{4} + \frac{36 \cdot x}{2} + \frac{36 \cdot 5\,x}{9} = 36 \cdot 65.$ (2)

By V, each denominator is cancelled by the factor 36,

$$9 \cdot 3\,x + 18\,x + 4 \cdot 5\,x = 36 \cdot 65. \qquad (3)$$

By III, $27\,x + 18\,x + 20\,x = 36 \cdot 65.$ (4)

By I, $65\,x = 36 \cdot 65.$ (5)

By D, V, $x = \frac{36 \cdot 65}{65} = 36.$ (6)

Check. Substitute $x = 36$ in (1).

EXERCISES AND PROBLEMS

Solve the following equations, indicating the principles used at each step. Check each solution by substituting in the original equation the value obtained. Translate each into a problem.

For instance, from equation 2: Find a number such that when increased by its half, its third, and its fourth, the sum is 25.

1. $\frac{n}{2}+\frac{n}{3}=5.$

2. $n+\frac{n}{2}+\frac{n}{3}+\frac{n}{4}=25.$

3. $\frac{n}{4}+4n-\frac{5n}{3}=\frac{3n}{2}+26.$

4. $\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+\frac{n}{10}=82.$

5. $7x+\frac{4x}{7}+\frac{3x}{5}+23=\frac{x}{5}+\frac{4x}{10}+5x+113.$

6. $4n+\frac{6n}{7}=\frac{3n+2}{2}+46.$

7. $12+\frac{4(9x+6)}{3}-\frac{2(3+11x)}{5}=\frac{5(4x+4)}{3}+14.$

8. $13b+7-\frac{9b+8}{7}=\frac{2b+9}{5}+38.$

9. $\frac{5a+7}{2}+\frac{2a+4}{3}=\frac{3a+9}{4}+5.$

10. $\frac{17a-5}{3}-\frac{10a+2}{4}=\frac{5a+7}{2}-5.$

11. $\frac{5x+3}{2}+\frac{5(2x+10)}{5}=2x+24.$

12. The sum of two numbers is 12, and the first number is $\frac{1}{3}$ as great as the second. What are the numbers?

13. The smaller of two numbers is $\frac{3}{8}$ of the larger. If their sum is 66, what are the numbers?

14. Find two consecutive integers such that 4 times the first minus 3 times the second equals 9.

15. Find three consecutive integers such that $\frac{1}{3}$ of the first plus the second minus $\frac{1}{2}$ the third equals 5.

16. Find three consecutive integers such that 3 times the first plus 9 times the second minus 4 times the third equals 73.

17. There are three numbers such that the second is 4 more than 9 times the first, and the third is 2 more than 6 times the first. If $\frac{1}{8}$ of the third is subtracted from $\frac{1}{7}$ of the second, the remainder is 3. Find the numbers.

18. There are three numbers such that the second is 2 more than 9 times the first and the third is 7 more than 11 times the first. The remainder when 4 times the third is subtracted from 13 times the second is 144. Find the numbers.

19. The population of Philadelphia (estimate of Census Bureau, 1904) was 508 thousand less than 20 times that of Dayton, Ohio. The population of St. Louis was 245 thousand more than 4 times that of Dayton. One-half the population of Philadelphia plus $\frac{1}{5}$ that of St. Louis was 821 thousand. Find the population of each city.

20. The shortest railway route from Boston to Chicago is 166 miles more than 4 times that from Boston to New York; and the shortest route from Boston to Atlanta is 196 miles less than 6 times that from Boston to New York. The distance from Boston to Chicago is 481 miles more than $\frac{1}{2}$ the distance from Boston to Atlanta. Find each of the three distances.

Solution. Let d = distance in miles from Boston to New York;
then $4d + 166$ = distance in miles from Boston to Chicago,
and $6d - 196$ = distance in miles from Boston to Atlanta,
and $4d + 166 = \frac{6d - 196}{2} + 481.$

By VI, $4d + 166 = 3d - 98 + 481.$

By *S*, II, $d = 481 - 98 - 166 = 217$ = distance from Boston to New York.

$4d + 166 = 1034$ = distance from Boston to Chicago,
and $6d - 196 = 1106$ = distance from Boston to Atlanta.

21. The railway mileage in the United States in 1900 was 8 thousand greater than twice that of 1880, and in 1904 it was 66 thousand less than 3 times that of 1880. One-half the mileage of 1900 plus $\frac{1}{3}$ that of 1904 equals 168 thousand. Find the railway mileage in 1880, 1900, and 1904.

22. The number of newspapers in the United States in 1880 was 381 less than 4 times that in 1850. The number in 1905 was 3700 more than twice that of 1880 and 7990 more than 6 times that of 1850. Find the number of newspapers in 1850, 1880, and 1905.

23. The number of telephones in the United States in 1900 was 40 thousand less than 30 times as great as the number in 1880. Half the number in 1905 was 660 thousand more than that in 1900 and 80 thousand more than 40 times that in 1880. Find the number of telephones in use in 1880, 1900, and 1905.

24. The displacement of the battleship *Kearsarge* is 1232 tons greater than that of the *Oregon*, and the displacement of the *Connecticut* is 4480 tons greater than that of the *Kearsarge*. One-third the displacement of the *Kearsarge* minus $\frac{1}{5}$ that of the *Connecticut* equals 640. Find the displacement of each ship.

25. The distance of the fixed star Vega from the sun is 11.7 light-years greater than the distance from the sun to Sirius. The distance of Regulus is 8 light-years less than twice that of

Vega. What is the distance of each star from the sun if $\frac{1}{2}$ the distance of Vega minus $\frac{1}{8}$ that of Regulus is 6 light-years?

Light travels at the rate of 186,000 miles in one second. A light-year is the distance traveled by light in one year.

26. The distance from the sun to the fixed star 61 Cygni is 4 light-years more than that from the sun to Alpha Centauri (the star nearest the sun), and the distance from the sun to Polaris (the pole star) is 6 times that of 61 Cygni. One-eighth the distance of Polaris minus $\frac{1}{8}$ the sum of the distances of the other two equals 2 light-years. How many light-years is each star from the sun?

27. In 1905 the wheat crop of Russia was 47 million bushels less than 4 times that of Argentina, and the crop of the United States was 77 million bushels less than 5 times that of Argentina. The crop of the United States exceeded that of Russia by 124 million bushels. What was the wheat crop of each country?

28. The total imports of the United States in 1900 were 60 million dollars less than 10 times the imports in 1800. The imports in 1906 were 473 million less than twice the imports of 1900, and 135 million more than 12 times the imports of 1800. Find the imports of the United States in 1800, 1900, and 1906.

29. The total exports of the United States in 1900 were 26 million dollars less than 20 times the exports in 1800. The exports in 1906 exceeded those of 1900 by 350 million and were 31 million less than 25 times those of 1800. Find the exports of 1800, 1900, and 1906.

30. The gross tonnage of the German merchant marine was (1906) 548 thousand more than that of the United States, and the tonnage of the British marine was 2662 thousand greater than 4 times that of the German marine. The British marine was greater by 930 thousand tons than twice that of Germany and 3 times that of the United States combined. Find the tonnage of each marine.

31. The population of San Francisco (estimate of Census Bureau, 1904) was 289 thousand more than that of Springfield, Massachusetts, and the population of Chicago was 132 thousand more than 5 times that of San Francisco. The population of Springfield minus $\frac{1}{28}$ that of Chicago was 2 thousand. Find the population of each city.

32. The per capita tax of New York City (1905) was $.49 less than twice that of Chicago and the per capita tax of Boston was $3.56 less than 3 times that of Chicago. The combined per capita tax of Boston and New York was $52.15. What was the per capita tax of each city?

33. The number of pupils attending public high schools in Chicago (1905) was 1237 more than twice as great as in Philadelphia. One-third the number of pupils in Chicago minus $\frac{1}{4}$ the number in Philadelphia was 2524. How many pupils attended public high schools in each city?

34. The number of pupils in public high schools in Boston (1905) was 1574 less than 4 times as many as in Baltimore. Three times the number in Boston minus 6 times the number in Baltimore equals 5268. How many pupils in each city?

35. The distance from San Francisco to Yokohama is 2031 statute miles less than 3 times that from San Francisco to Honolulu, and the distance from San Francisco to Hongkong is 2219 more than twice that from San Francisco to Honolulu. One-fifth the distance to Hongkong plus $\frac{1}{8}$ the distance to Yokohama equals 3152. Find the distance from San Francisco to each of the other three cities.

36. The distance from New York to Singapore is 542 statute miles more than 3 times that from New York to London, and the distance from New York to Manila is 5342 miles less than 5 times that from New York to London. Three times the distance from New York to Manila plus 6974 miles equals 4 times the distance from New York to Singapore. Find the distance from New York to each of the other three cities.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

42. Thus far the numbers used have been precisely the same as in arithmetic, though their representation by means of letters and some of the methods used in operating upon them are peculiar to algebra.

43. A new kind of number will now be studied in connection with problems of the following type.

Illustrative Problems. 1. If a man gains \$1500, and then loses \$800, what is the net result? *Answer*, \$700 gain.

2. The assets of a commercial house are \$250,000, and the liabilities are \$275,000. What is the net financial status of the house? *Answer*, \$25,000 net liabilities.

3. The thermometer rises 18 degrees and then falls 28 degrees. What direct change in temperature would produce the same result? *Answer*, 10 degrees fall.

4. A man travels 700 miles east and then 400 miles west. What direct journey would bring him to the same final destination? *Answer*, 300 miles east.

In the statement of each of these problems the numbers are applied to things which are *opposite in quality;* namely, gain and loss, assets and liabilities, rise and fall, distances measured in two opposite directions, as east and west.

The answer in each case not only gives the proper arithmetical number, but also connects with this number one of the two opposite qualities involved in the problem.

E.g. The answer in (1) is \$700 *gain* and not simply \$700; in (2) it is not \$25,000, but \$25,000 *liabilities;* in (3) it is not 10°, but 10° *fall;* and in (4) it is not 300 miles, but 300 miles *east.*

These problems are illustrations of the sense in which the word "opposite" is here used.

E.g. Gain and loss are opposite in that they *annul* each other when taken together; rise and fall of temperature are opposite in that one *counteracts* the other.

44. In all problems involving one or both of two qualities which are opposite in this sense there is constant need to distinguish which one is meant.

E.g. On a day in winter it is not sufficient to say the temperature is 5°; we must specify whether it is *above* or *below* zero.

In the case of the thermometer we call the temperature positive if it is above zero and negative if it is below zero.

These words *positive* and *negative* are used to describe all pairs of qualities which are opposite in the sense here understood.

The signs $^+$ and $^-$ stand respectively for the words "positive" and "negative."

E.g. $\$^+700$ is read *positive* \$700, and means either \$700 gain or \$700 assets, according to the problem in which it occurs. Likewise $\$^-700$ is read *negative* \$700, and means either \$700 loss or \$700 liability.

Referring to the thermometer, $^+18°$ is read positive 18°, and means either a temperature of 18° *above* zero or a *rise* of the mercury 18° from any point. The latter is the meaning in Problem 3. Similarly $^-28°$ means either a temperature 28° *below* zero or a *fall* of the mercury 28° from any point.

It is commonly agreed to call gain positive and loss negative, assets positive and liabilities negative, above zero positive and below zero negative, motion upward positive and downward negative, motion to the right positive and to the left negative.

45. Numbers marked with the sign $^+$ are called **positive** and those marked with the sign $^-$ are called **negative**.

The positive sign may be omitted; that is, a number with neither sign written is understood to be positive.

ADDITION OF POSITIVE AND NEGATIVE NUMBERS

46. While in each of the problems on page 47 the result was obtained by subtracting one number from the other, yet they are not properly subtraction problems, but addition problems.

E.g. In Problem 1, we are not asking for the *difference* between \$1500 gain and \$800 loss, but for the *net result* when the gain and the loss are taken together; that is, the *sum of the profit and loss.* Hence we say \$1500 gain + \$800 loss = \$700 gain, or, using positive and negative signs,

$$^{+}1500 + {}^{-}800 = {}^{+}700.$$

Similarly in Problem 2,

\$250,000 assets + \$275,000 liabilities = \$25,000 net liabilities,

Or $$^{+}250{,}000 + {}^{-}275{,}000 = {}^{-}25{,}000.$$

In Problem 3, 18° rise + 28° fall = 10° fall,

Or $$^{+}18 + {}^{-}28 = {}^{-}10.$$

In Problem 4,

700 miles east + 400 miles west = 300 miles east,

Or $$^{+}700 + {}^{-}400 = {}^{+}300.$$

NOTE.—In the last case east is called $^{+}$ and west $^{-}$, since, as a map is usually held, east is to the right and west to the left. Likewise north or up is called $^{+}$ and down or south is called $^{-}$.

PROBLEMS

1. A balloon which exerts an upward pull of 460 pounds is attached to a car weighing 175 pounds. What is the net upward or downward pull? Express this as a problem in addition, using positive and negative numbers.

Solution. 460 lb. upward pull + 175 lb. downward pull = 285 lb. net upward pull. Using positive numbers to represent upward pull and negative numbers to represent downward pull, this equation becomes

$$^{+}460 + {}^{-}175 = {}^{+}285.$$

2. How would the morning paper print the following thermometer readings from the weather report? Jacksonville 38° above zero, Seattle 5° below zero, Nashville 28° above zero, Chicago 13° below zero.

3. The temperature rises 15° and then falls 24°. What direct change in temperature would produce the same final result? Express this as an example in addition.

4. A vessel on the equator sailed north 3° and was then forced south 5° by a hurricane. It then resumed its course northward 8°. Express this as a sum and show the final position with reference to the equator. (See note, § 46.)

In each of the following translate the solution into the language of algebra by means of positive and negative numbers, as in Problem 1.

5. A 450-pound weight is attached to a balloon which exerts an upward pull of 600 pounds. What is the net upward or downward pull?

6. A man's property amounts to $45,000 and his debts to $52,000. What is his net debt or property?

7. The assets of a bankrupt firm amount to $245,000 and the liabilities to $325,000. What are the net assets or liabilities?

8. A weight exerting a downward pull of 280 pounds when submerged in water is attached to a float which will just support a weight of 240 pounds. What is the net pressure upward or downward when both are submerged?

9. A man on the deck of a steamer is walking at the rate of 4 miles an hour toward the stern. If the boat is sailing eastward in a river at the rate of fifteen miles per hour, what is the actual motion of the man with respect to the bank of the river?

10. A man can row a boat at the rate of 6 miles per hour. How fast can he proceed against a stream flowing at the rate of $2\frac{1}{2}$ miles per hour; 7 miles per hour?

11. A steamer which can make 12 miles per hour in still water is running against a current flowing 15 miles per hour. How fast and in what direction does the steamer move?

12. A dove capable of flying 40 miles per hour in calm weather is flying against a hurricane blowing at the rate of 60 miles per hour. How fast and in what direction is the dove moving?

13. If of two partners, one loses \$1400 and the other gains \$3700, what is the net result to the firm?

14. A man's income is \$2400 and his expenses \$1500 per year. What is his saving?

15. A man loses \$800 and then loses \$600 more. What is the combined loss? Indicate the result as the sum of two negative numbers.

16. A man gains \$500 and then gains \$700 more. What is the combined gain? Express the result as the sum of two positive numbers.

17. A tug which can steam 9 miles per hour in still water is going down a stream whose current is 6 miles per hour. How fast is the tug moving?

18. The thermometer falls 8° and then 17° more. Express the combined result of the two changes as a negative number.

47. Definitions. Positive and negative numbers are sometimes called **signed numbers**, because each such number consists of an arithmetic part, together with **a sign of quality**.

The arithmetic part of a signed number is called its **absolute value**.

Thus, the absolute value of $^{+}3$ and also of $^{-}3$ is 3. Two signed numbers are of *like* quality when they have the same signs, and of *opposite* quality when one is positive and the other negative.

In the one case we say they have *like signs*, in the other opposite or *unlike signs*. The preceding exercises illustrate the following principle:

48. Principle IX. *To add two signed numbers of like quality, find the sum of their absolute values, and prefix to this the common sign of quality.*

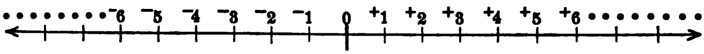
To add two signed numbers of opposite quality, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case their absolute values are equal, their sum is zero.

The sum of two signed numbers thus obtained is called their **algebraic sum.**

49. This principle may be further illustrated by means of the following **graphic representation** of signed numbers.

On an unlimited straight line call some starting point zero, and lay off from this point equal divisions of the line indefinitely both to the right and to the left, as shown in the figure.



In order to describe the position of any one of these division points, we require not only an integer of arithmetic, to specify *how far* to count from the starting point (the point marked zero) in order to reach the given point, but also a *quality sign*, to indicate the *direction* of the counting.

E.g. + 7 marks the division point 7 units to the right of zero, and −5 marks the point 5 units to the left of zero. Such a diagram is called the **scale of signed numbers.**

The part of the scale to the right, taken alone, would need no signs, and would picture the integral numbers of arithmetic, while the

two parts together require the distinguishing signs of quality, and picture the integral numbers of algebra.

Fractions would of course be pictured at points between the integral division points, on the right or the left of the scale, according as the fractions are positive or negative.

ADDITION BY COUNTING

50. In arithmetic two numbers are added by starting with either and counting forward as many units as there are in the other.

E.g. To add 3 and 5 we start with 5 on the number scale and count 6, 7, 8; or start with 3 and count 4, 5, 6, 7, 8.

51. In algebra two signed numbers are added in the same manner except that the *direction*, forward or backward, in which we count, is determined by the sign, $^+$ or $^-$, of the number which we are adding.

Thus, to add $^+7$ and $^+5$, begin at 7 to the right of the zero point in the scale of signed numbers and count 5 more toward the right, or begin at 5 to the right and count 7 more to the right, in either case arriving at $^+12$.

To add $^-7$ and $^-5$, we may begin at 7 to the left and count 5 more toward the left, or begin at 5 to the left and count 7 more in that direction, in either case arriving at $^-12$.

To add $^+7$ and $^-5$, begin at 7 to the right and count 5 toward the left, or begin at 5 to the left and count 7 toward the right, in either case arriving at $^+2$.

To add $^-7$ and $^+5$, begin at 7 to the left and count 5 toward the right, or begin at 5 to the right and count 7 toward the left, in either case reaching $^-2$.

I.e. $^+7 + {}^+5 = {}^+12$, $^-7 + {}^-5 = {}^-12$, $^+7 + {}^-5 = {}^+2$, $^-7 + {}^+5 = {}^-2$.

52. In adding several signed numbers, we reach the same result whether we add them in the order in which they happen to be given, or in any other order. Thus we may add first all

the positive numbers and then all the negative numbers, and finally combine these two results.

E.g. $^{+}5+{}^{-}8+{}^{+}7+{}^{-}6$, taken in order from left to right, gives $^{-}3$ as the sum of the first two, $^{+}4$ as the sum of the first three, and $^{-}2$ as the final result. But $^{+}5+{}^{+}7={}^{+}12$, $^{-}8+{}^{-}6={}^{-}14$, and $^{+}12+{}^{-}14={}^{-}2$, the same result as before.

53. Addition by counting makes it clear that two numbers of opposite signs tend to cancel each other when added.

E.g. In adding $^{-}5$ to $^{+}8$ we start with $^{+}8$, which we reach by counting from zero eight units to the right, and then add $^{-}5$ by counting five units to the left, thus retracing five of the units just counted. That is, $^{-}5$ annuls five of the $^{+}8$ units, leaving the sum $^{+}3$.

In case two numbers have opposite signs and equal absolute values one completely cancels the other.

E.g. $^{+}8+{}^{-}8=0$.

EXERCISES AND PROBLEMS

Perform the following indicated additions:

1. $^{-}31+{}^{+}42$. 3. $^{+}68+{}^{-}46$. 5. $^{+}104+{}^{-}245$.

2. $^{-}17+{}^{-}13$. 4. $^{+}34+{}^{-}46$. 6. $^{-}11\frac{3}{4}+{}^{+}8\frac{1}{4}$.

7. $^{+}16+{}^{-}3+{}^{+}7+{}^{+}4+{}^{-}19$. 8. $^{+}42+{}^{+}74+{}^{-}92+{}^{-}7+{}^{-}3$.

9. $^{-}13\,n+{}^{+}7\,n={}^{+}7\,n+{}^{-}13\,n={}^{-}6\,n$.

Solution. By Principle IX, to add $^{-}13\,n$ to $^{+}7\,n$ or to add $^{+}7\,n$ to $^{-}13\,n$ is to "find the difference of their absolute values and prefix to this the sign of that one whose absolute value is the greater." That is, $^{-}13\,n+{}^{+}7\,n={}^{-}(13\,n-7\,n)={}^{-}6\,n$, since by Principle II, $13\,n-7\,n=6\,n$.

10. $^{+}14\,t+{}^{-}8\,t+{}^{-}3\,t+{}^{-}45\,t$. 11. $^{+}68x+{}^{+}34x+{}^{-}16x+{}^{-}3x$.

12. $296\,rt+{}^{-}367\,rt+{}^{-}119\,rt$. 13. $^{-}3(a+b)+{}^{+}4(a+b)$.

14. $^{+}7\,(x-y)+{}^{-}5\,(x-y)$. 15. $^{-}81(r+s)+{}^{-}91(r+s)$.

16. A man travels 10 miles east, then 23 miles west, and finally 5 miles east. Express his final *distance* and *direction* from the starting point as the sum of three signed numbers.

17. The temperature falls 35°, rises 24°, falls 3°, rises 17°, and finally falls 5°. What is the net change in temperature between the last reading and the original reading?

18. A real estate firm gains \$5000 on one transaction, loses \$2500 on a second, loses \$1400 on a third, and gains \$200 on a fourth. What is the aggregate result of the four transactions?

AVERAGES OF SIGNED NUMBERS

54. Half the sum of two numbers is called their average. Thus 6 is the average of 4 and 8. Similarly, the average of three numbers is one-third of their sum, and in general the average of n numbers is the sum of the numbers divided by n.

Find the average of each of the following sets:

1. 10, 12, 14, 16, 18.
2. 5, 9, 20, 30, 3.
3. 11, 10, 4, 6, 5.
4. 7, 9, 11, 13, 15.
5. 7, 10, 21, 29, 30.
6. 13, 8, 9, 10, 4.

The average gain or loss per year for a given number of years is the *algebraic sum* of the yearly gains and losses divided by the number of years.

Illustrative Problem. A man lost \$400 the first year, gained \$300 the second, and gained \$1000 the third. What was the average loss or gain?

Solution. $\frac{{}^{-}400 + {}^{+}300 + {}^{+}1000}{3} = \frac{{}^{+}900}{3} = {}^{+}300$

That is, the average gain is \$300.

Illustrative Problem. During four years a certain business shows an average annual gain of \$5000. What was the loss or gain the first year, if for the remaining years there were gains of \$8000, \$9000, and \$7500 respectively?

Solution. Let x represent the number of dollars gained or lost the first year. Then the average for the four years is

$$\frac{x + {}^{+}8000 + {}^{+}9000 + {}^{+}7500}{4}.$$

But the average for the four years is given as \$5000.

Hence $\dfrac{x + {}^{+}8000 + {}^{+}9000 + {}^{+}7500}{4} = {}^{+}5000.$

By M, $x + {}^{+}8000 + {}^{+}9000 + {}^{+}7500 = {}^{+}20000.$

By IX, $x + {}^{+}24500 = {}^{+}20000.$

By A, $x + {}^{+}24500 + {}^{-}24500 = {}^{+}20000 + {}^{-}24500.$

By IX, $x = {}^{-}4500.$

Hence there was a loss of \$4500 the first year.

PROBLEMS

1. Find the average of \$1800 loss, \$3100 loss, \$6800 gain, \$10,800 loss, and \$31,700 gain.

2. Find the average of \$180 gain, \$360 loss, \$480 loss, \$100 gain, \$700 gain, \$400 gain, \$1300 loss, \$300 gain, \$4840 gain, and \$12,000 gain.

3. A merchant gained an average of \$2800 per year for 5 years. The first year he gained \$3000, the second \$1500, the third \$4000, and the fourth \$2400. Did he gain or lose and how much during the fifth year?

4. A certain business shows an average gain of \$4000 per year for 6 years. During the first 5 years the results were, \$8000 loss, \$10,000 gain, \$7000 gain, \$3000 gain, and \$12,000 gain. Find the loss or gain during the sixth year.

5. Find the average of the following temperatures: 7 A.M., ${}^{-}4°$; 8 A.M., ${}^{-}2°$; 9 A.M., ${}^{-}1°$; 10 A.M., ${}^{+}1°$; 11 A.M., ${}^{+}5°$; 12 M., ${}^{+}7°$.

6. During the 12 hours ending at 6 A.M., January 19, 1892, the U. S. Weather Bureau at Helena, Montana, recorded the following temperatures: ${}^{-}9°$, ${}^{-}8°$, ${}^{-}8°$, ${}^{-}9°$, ${}^{-}9°$, ${}^{-}9°$, ${}^{-}8°$, ${}^{+}36°$, ${}^{+}37°$, ${}^{+}40°$, ${}^{+}20°$, ${}^{+}16°$. Find the average temperature for the 12 hours.

Find the average yearly temperatures at the following places, the monthly averages having been recorded as given below:

7. For New York City: $^+29°$, $^+33°$, $^+39°$, $^+46°$, $^+53°$, $^+63°$, $^+67°$, $^+67°$, $^+61°$, $^+52°$, $^+47°$, $^+41°$.

8. For Singapore, Straits Settlement: $^+81°$, $^+84°$, $^+85°$, $^+85°$, $^+85°$, $^+86°$, $^+86°$, $^+87°$, $^+85°$, $^+84°$, $^+83°$, $^+81°$.

9. For St. Vincent, Minnesota: $^-5°$, $0°$, $^+15°$, $^+35°$, $^+55°$, $^+60°$, $^+66°$, $^+63°$, $^+55°$, $^+40°$, $^+22°$, $^+5°$.

10. For Nerchinsk, Siberia: $^-23°$, $^-13°$, $^-10°$, $^+35°$, $^+55°$, $^+70°$, $^+70°$, $^+64°$, $^+50$, $^+30°$, $^+5°$, $^-15°$.

If the latitude of a place is midway between the latitudes of two given places, then its latitude equals one-half the *algebraic sum* of the two given latitudes.

Thus, if the latitude of one place is $^+16°$ and that of the other $^-56°$, then the latitude of the place midway between them is $\frac{^-56° + {}^+16°}{2} = \frac{^-40°}{2} = {}^-20°$.

The data used in some of the following problems vary slightly from the published records, but in no case more than $5'$.

11. The latitude of New Orleans, Louisiana, is $^+30°$, and of Toronto, Canada, $^+43°\ 40'$. Find the latitude of Norfolk, Virginia, which is midway between these latitudes. (Positive numbers represent north latitude and negative numbers represent south latitude.)

12. The latitude of Alexandria, Egypt, is $^+31°\ 10'$, and of Christiania, Norway, $^+59°\ 50'$. That of Venice, Italy, is midway between these latitudes. Find the latitude of Venice.

13. The longitude of Edinburgh, Scotland, is $^-3°\ 10'$, and of Warsaw, Poland, $^+21°$. Find the longitude of Bremen, Germany, which is midway between these. (Positive numbers represent east longitude and negative numbers west longitude.)

14. The longitude of Vienna, Austria, is $^{+}16° 20'$, and of Splügen Pass, Switzerland, $^{+}9° 20'$. The longitude of Splügen Pass is midway between the longitudes of Vienna and Paris, France. Find the longitude of Paris.

15. The longitude of Brussels, Belgium, is $^{+}4° 20'$, and of Jena, Germany, $^{+}11° 40'$. The longitude of Brussels is midway between those of Jena, and Liverpool, England. Find the longitude of Liverpool.

16. The longitude of Berlin, Germany, is $^{+}13° 20'$, and of St. Petersburg, Russia, $^{+}30° 20'$. The longitude of Berlin is midway between those of St. Petersburg and Madrid, Spain. Find the longitude of Madrid.

17. The latitude of Rome, Italy, is $^{+}41° 55'$, and of Cuxhaven, Germany, $^{+}53° 55'$. The latitude of Rome is midway between those of Cuxhaven and Cairo, Egypt. Find the latitude of Cairo.

18. The longitude of Calais, France, is $^{+}1° 55'$ and of Lucknow, India, $^{+}80° 55'$. The longitude of Calais is midway between those of Lucknow and Washington, D.C. What is the longitude of Washington?

SUBTRACTION OF SIGNED NUMBERS

55. In arithmetic the accuracy of subtraction is tested by showing that the remainder added to the subtrahend equals the minuend.

E.g. We say $8 - 5 = 3$ because $5 + 3 = 8$.

Indeed, we may, and often do perform subtraction by starting with the subtrahend and counting until we reach the minuend.

E.g. A clerk in changing a dollar bill after a purchase of 63 cents might count out two pennies, a dime, and a quarter, saying "65, 75, one dollar." That is, he counts from 63 cents (the subtrahend) up to 100 cents (the minuend).

56. In like manner, signed numbers also may be subtracted, but we must find not only the *number of units, but also the direction* from the subtrahend to the minuend. This is most easily done by reference to the number scale, page 52.

Ex. 1. $^{+}8 - {}^{+}5 = {}^{+}3$, since in passing from $^{+}5$ to $^{+}8$ on the number scale we count 3 units in the *positive* direction.

Ex. 2. $^{-}8 - {}^{-}5 = {}^{-}3$, since from $^{-}5$ to $^{-}8$ we count 3 units in the *negative* direction.

Ex. 3. $^{-}8 - {}^{+}5 = {}^{-}13$, since from $^{+}5$ to $^{-}8$ we count 13 units in the *negative* direction.

Ex. 4. $^{+}8 - {}^{-}5 = {}^{+}13$, since from $^{-}5$ to $^{+}8$ we count 13 units in the *positive* direction.

57. In each of these examples we see that the result fulfills the test of arithmetic; namely,

$$\textit{Subtrahend} + \textit{Difference} = \textit{Minuend}$$

Hence $^{-}8 - {}^{-}5 = {}^{-}3$ is checked by finding that $^{-}5 + {}^{-}3 = {}^{-}8$.
$^{+}8 - {}^{-}5 = {}^{+}13$ is checked by finding that $^{-}5 + {}^{+}13 = {}^{+}8$.

EXERCISES AND PROBLEMS

Perform the following indicated subtractions by finding the *distance* and *direction* on the number scale from subtrahend to minuend, and apply the check to each result.

1. $^{-}10 - {}^{-}5$	6. $^{-}17 - {}^{-}20$	11. $^{+}93 - {}^{+}22$
2. $^{-}15 - {}^{+}5$	7. $^{+}6 - {}^{-}14$	12. $^{+}17 - {}^{-}13$
3. $^{+}20 - {}^{-}15$	8. $^{+}7 - {}^{-}9$	13. $^{-}78 - {}^{-}37$
4. $^{+}11 - {}^{+}3$	9. $^{-}11 - {}^{+}6$	14. $^{+}57 - {}^{+}84$
5. $^{-}11 - {}^{+}5$	10. $^{-}21 - {}^{-}6$	15. $^{-}48 - {}^{-}31$

16. How many degrees, and in what direction, must the temperature change in order to vary from 12° below zero to 38° above zero? This is an example in subtraction, since we are required to find a number which added to $^{-}12°$ gives $^{+}38°$. Hence we write it $^{+}38° - {}^{-}12° =$ what?

17. How many degrees, and in what direction, does the thermometer change in passing from 27° above zero to 3° below zero? That is, $^-3° - ^+27° =$ what?

18. What must be added to \$35 loss to make the sum \$30 gain? That is, $^+30 - ^-35 =$ what?

19. What must be added to \$15 gain to make the sum \$8 loss? That is, $^-8 - ^+15 =$ what?

58. Subtraction always possible. In arithmetic subtraction is possible only when the subtrahend is less than or equal to the minuend.

E.g. 5 from 8 leaves 3, 5 from 5 leaves 0; but we cannot take 5 from 2, since when we have subtracted 2 units from 2 there are no more to take away.

However, by means of negative numbers we can as easily perform the subtraction, 2 minus 5, as 5 minus 2.

Thus, $2 - 5 = ^-3$, since $^-3 + 5 = 2$; and $5 - 2 = ^+3$, since, $^+3 + 2 = 5$.

It thus appears that, in terms of signed numbers, $a - b$ has a meaning no matter what numbers are represented by a and b; that is, $a - b$ means the number which added to b gives a.

59. A short rule for subtraction. Since $^+8 - ^-5 = ^+13$, and since $^+8 + ^+5 = ^+13$, it follows that subtracting $^-5$ from $^+8$ gives the same result as adding $^+5$ to $^+8$. Similarly $^-8 - ^-5 = ^-3$ and $^-8 + ^+5 = ^-3$.

Hence *subtracting a negative* number is equivalent to *adding a positive* number of the same absolute value.

Since $^+8 - ^+5 = ^+3$, and since $^+8 + ^-5 = ^+3$, it follows that subtracting $^+5$ from $^+8$ gives the same result as adding $^-5$ to $^+8$. Similarly $^-8 - ^+5 = ^-13$ and $^-8 + ^-5 = ^-13$.

Hence *subtracting a positive* number is equivalent to *adding a negative* number of the same absolute value.

These statements are illustrated by such facts as: *Removing a debt* is equivalent to *adding property* and *removing property* is equivalent to *adding debt.*

Perform the following subtractions as explained in this paragraph, by changing the sign of the subtraheud and adding:

1. $^-5 - {}^-2$.	5. $^+57 - {}^-32$.	9. $^+37 - {}^+50$.
2. $^-4 - {}^+1$.	6. $^-32 - {}^+34$.	10. $^-23 - {}^+57$.
3. $^-5 - {}^+2$.	7. $^-52 - {}^-32$.	11. $^-16\,a - {}^+4\,a$
4. $^+3 - {}^-5$.	8. $^-16 - {}^-12$.	12. $^+13\,t - {}^-20\,t$

The preceding exercises illustrate the following principle:

60. Principle X. *To subtract one signed number from another signed number, add the subtrahend with its sign changed to the minuend.*

The change in the sign of the subtrahend may be made *mentally* without re-writing the problem. The results are to be checked by showing that the difference added to the subtrahend equals the minuend.

EXERCISES

1. From $^+6 + {}^-2$ subtract $^-14$.
2. From $^-6\,a + {}^+2a$ subtract $^-14\,a$.
3. From $17\,ab + 8\,ab$ subtract $^-35\,ab$.
4. From $5\,ax + 4\,ax$ subtract $7\,ax + 2\,ax$.
5. From $54\,abc + {}^-47\,abc + 36\,abc$ subtract $80\,abc$.
6. From $54 \cdot 13 + {}^-47 \cdot 13 + 36 \cdot 13$ subtract $80 \cdot 13$.
7. From $29 \cdot 3 \cdot 11 + 37 \cdot 3 \cdot 11$ subtract $^-34 \cdot 3 \cdot 11$.
8. From $29\,xy + 37\,xy$ subtract $^-34\,xy$.
9. Solve $x + 8 = 4$.

Solution. Subtract $^+8$ from each member (which is equivalent to adding $^-8$).

Then $x = {}^{+}4 - {}^{+}8 = {}^{-}4$, which is correct, since ${}^{-}4 + {}^{+}8 = {}^{+}4$. This is a problem in subtraction, since one of two numbers, 8, and their sum, 4, are given, and we are to find the second number, which is represented by x.

Solve the following equations:

10. $x + {}^{-}3 = 7.$	16. $x + 9 = 3.$
11. $x + {}^{-}9 = 1.$	17. ${}^{-}4 + x = 7.$
12. $3 + x = 0.$	18. ${}^{-}5 + a = 4.$
13. $x + {}^{-}1 = 2.$	19. ${}^{-}20 + t = {}^{-}12.$
14. $x + 13 = 7.$	20. $8 + n = {}^{-}15.$
15. $x + 4 = 2.$	21. $k + {}^{-}40 = {}^{-}65.$

MULTIPLICATION OF SIGNED NUMBERS

61. The multiplication of signed numbers is illustrated by the following problems.

Illustrative Problem. A balloonist, just before starting, makes the following preparations: (*a*) He adds 9000 cubic feet of gas with a lifting power of 75 pounds per thousand cubic feet; (*b*) He takes on 8 bags of sand, each weighing 15 pounds. How does each of these operations affect the buoyancy of the balloon?

Solution. (*a*) A lifting power of 75 lbs. is indicated by ${}^{+}75$, and adding such a power 9 times is indicated by ${}^{+}9$. Hence ${}^{+}9 \cdot {}^{+}75 = {}^{+}675$, the total lifting power added.

(*b*) A weight of 15 lbs. is indicated by ${}^{-}15$, and adding 8 such weights is indicated by ${}^{+}8$. Since the total weight added is 120 lbs., we have ${}^{+}8 \cdot {}^{-}15 = {}^{-}120$.

Illustrative Problem. During the course of his journey this balloonist opens the valve and allows 2000 cubic feet of gas to escape, and later throws overboard 4 bags of sand. What effect does each of these operations produce on the balloon?

Solution. (*a*) The gas, being a lifting power, is positive, but the removal of 2000 cubic feet of it is indicated by $^-2$, and the result is a depression of the balloon by 150 lbs.; that is, $^-2 \cdot {}^+75 = {}^-150$.

(*b*) The removal of 4 weights is indicated by $^-4$, but the weights themselves have the negative quality of downward pull. Hence to remove 4 weights of 15 lbs. each is equivalent to increasing the buoyancy of the balloon by 60 lbs; that is, $^-4 \cdot {}^-15 = {}^+60$.

62. These illustrations of multiplying signed numbers are natural extensions of the process of multiplication in arithmetic.

E.g. Just as $3 \cdot 4 = 4 + 4 + 4 = 12$, so $3 \cdot {}^-4 = {}^-4 + {}^-4 + {}^-4 = {}^-12$, and since $3 \cdot 4$ is the same as $^+3 \cdot {}^+4$, we write $^+3 \cdot {}^+4 = {}^+12$.

Again, just as we take the multiplicand *additively* when the multiplier is a positive integer, so we take it *subtractively* when the multiplier is negative integer.

E.g. $^-3 \cdot {}^+4$ means to subtract $^+4$ three times; that is, to subtract $^+12$. But to subtract $^+12$ is the same as to add $^-12$. Hence $^-3 \cdot {}^+4 = {}^-12$. Again, $^-3 \cdot {}^-4$ means to subtract $^-4$ three times; that is, to subtract $^-12$. But to subtract $^-12$ is the same as to add $^+12$. Hence $^-3 \cdot {}^-4 = {}^+12$.

EXERCISES AND PROBLEMS

Explain the following indicated multiplications and find the product in each case:

1. $^-3 \cdot {}^-10$.	5. $43 \cdot {}^-192$.	9. $71 \cdot {}^-x$.
2. $^-3 \cdot {}^+10$.	6. $^-27 \cdot {}^-235$.	10. $^-112 \cdot {}^-t$.
3. $^-5 \cdot {}^+50$.	7. $^-5 \cdot {}^+r$.	11. $^-14 \cdot y$.
4. $^-75 \cdot {}^-89$.	8. $^+16 \cdot {}^-r$.	12. $^-20 \cdot {}^-v$.

13. A man gained \$212 each month for 5 months, then lost \$175 per month for 3 months. Express his net gain or loss as the sum of two products.

14. A man gained $2100 during a certain year. For the first 4 months he lost $125 per month. During the next 5 months he gained $500 per month. Find his gain or loss during the remaining 3 months of the year. Express the net gain as the sum of two products.

15. A raft is made of cork and iron. What effects are produced upon its floating qualities by the following changes? (*a*) Adding 4 braces, each weighing (under water) 5 lbs. (*b*) Removing 3 pieces of cork, each capable of sustaining 3 lbs. (*c*) Adding 10 pieces of cork, each capable of sustaining 7 lbs.

16. What are the effects on a shipwrecked man's ability to float? (*a*) If he holds fast to 3 bags of gold, each weighing 10 lbs. (*b*) If he ties on two life preservers, each capable of supporting 15 lbs. (*c*) If he throws away his two boots, each weighing 2 lbs.

The preceding exercises illustrate the following principle:

63. Principle XI. *If two signed numbers are of the same quality, their product is positive; if they are of opposite quality, their product is negative. The absolute value of the product is the product of the absolute values of the factors.*

In applying this principle observe that the sign of the product is obtained quite independently of the absolute value of the two factors.

E.g. $\frac{3}{4} \cdot {}^{-}5 = {}^{-}(\frac{15}{4}) = {}^{-}3\frac{3}{4}$; ${}^{-}12 \cdot {}^{-}3.5 = {}^{+}42$.

64. Principle XI is also stated in symbols as follows:

$$\boldsymbol{{}^{+}a \cdot {}^{+}b = {}^{+}ab,\ {}^{-}a \cdot {}^{-}b = {}^{+}ab,\ {}^{+}a \cdot {}^{-}b = {}^{-}ab,\ {}^{-}a \cdot {}^{+}b = {}^{-}ab.}$$

The product of several signed numbers is found as illustrated in the following:

${}^{-}2 \cdot {}^{+}5 \cdot {}^{-}3 \cdot {}^{-}4 \cdot {}^{+}6 = {}^{-}10 \cdot {}^{-}3 \cdot {}^{-}4 \cdot {}^{+}6 = {}^{+}30 \cdot {}^{-}4 \cdot {}^{+}6 = {}^{-}120 \cdot {}^{+}6 = {}^{-}720$. That is, any two factors are multiplied together, then this product by another factor, etc., until all the factors are multiplied.

65. Evidently the factors in such a product may be taken in any desired order. Let the student try other orders in the above example.

Since the product of all positive factors is positive, the final sign depends upon the number of negative factors. If this number is *even*, the product is positive; if it is *odd*, the product is negative.

E.g. If there are 5 negative factors, the product is negative; if there are 6, it is positive.

EXERCISES

In the following exercises determine the sign of the product before finding its absolute value. State each principle used in the reduction to the final form.

1. ${}^-4 \cdot {}^+3 \cdot {}^-6 \cdot {}^-7$.
2. ${}^-50 \cdot {}^-20 \cdot {}^-30 \cdot {}^-40$.
3. ${}^-a \cdot {}^-b \cdot {}^+c \cdot {}^+d \cdot {}^+e$.
4. $a \cdot {}^-b \cdot {}^+c \cdot {}^-d \cdot {}^-x$.
5. ${}^-5({}^-3 + {}^-7)$.
6. $a({}^-b - {}^-c)$.
7. ${}^-c(x - {}^-y)$.
8. $7\,a(x + {}^-y - {}^-z)$.
9. $n(3\,a + {}^-4\,a - {}^+5\,a)$.
10. $8\,(16\,x + {}^-20\,x) \times 4$.
11. $5\,x - 3({}^-2\,x + {}^+3\,x - {}^-4\,x)$.
12. $6\,r + 4(3\,r - {}^-5\,r + {}^-7\,r)$.
13. ${}^-5\,({}^-4 \cdot {}^+3 \cdot {}^-2)$.
14. $6\,x - {}^-14\,x - ({}^-5\,x + {}^-7\,x)$.
15. $8\,y - 16\,y + (4\,y + {}^-11\,y)$.
16. $11\,t - 20\,(t + {}^-3 \cdot {}^-5\,t)$.

DIVISION OF SIGNED NUMBERS

66. In arithmetic we test the correctness of division by showing that the quotient multiplied by the divisor equals the dividend.

E.g. $27 \div 9 = 3$, because $9 \cdot 3 = 27$.

Hence **division** may be defined as the process of finding one of two factors when their product and the other factor are given.

This definition also applies to the division of signed numbers. In dividing signed numbers, however, we must determine the *sign* of the quotient as well as its *absolute value*.

E.g. $^-42 \div {}^+6 = {}^-7$, because $^-7 \cdot {}^+6 = {}^-42$;
also $^-42 \div {}^-6 = {}^+7$, because $^+7 \cdot {}^-6 = {}^-42$.

So in every case the test is:

Quotient $\times$ *Divisor* $=$ *Dividend.*

In like manner perform the following:

1. $^-25 \div 5$.
2. $^-ab \div a$.
3. $^+5xy \div {}^-x$.
4. $^-9rs \div {}^+3$.
5. $^+75y \div {}^-15$.
6. $^-121x \div {}^+11$.

The preceding exercises illustrate the following principle:

67. Principle XII. *The quotient of two signed numbers is positive if the dividend and divisor have the same sign, negative if they have opposite signs. The absolute value of the quotient is the quotient of the absolute values of dividend and divisor.*

EXERCISES

Perform the following indicated divisions. Check by multiplying quotient by divisor.

1. $^-28 \div {}^+7$.
2. $^-42 \div {}^-6$.
3. $51 \div {}^-17$.
4. $21xy \div 3$.
5. $^-16ab \div {}^-4$.
6. $^-15ax \div x$.
7. $^-32(a-b) \div {}^-(a-b)$.
8. $27(x+y) \div {}^-9$.
9. $4 \cdot {}^-9x \div {}^-3$.
10. $3 \cdot 8t \div {}^-4$.
11. $100 \cdot {}^-99x \div {}^-25$.
12. $1600 \cdot 87y \div {}^-400$.
13. $(8x+4y) \div {}^-2$.
14. $(16a + {}^-20b) \div {}^-4$.
15. $({}^-6r + 9s - {}^-12t) \div {}^-3$.
16. $(7ax + {}^-14ay - {}^-21bz) \div {}^+7$.
17. $(12xy - {}^-3ax) \div {}^-x$.
18. $(27ab - 36ac) \div {}^-9$.
19. $(3 \cdot 4y + 6 \cdot 8x) \div {}^-3$.
20. $2a(4x-3y-z) \div a$.

68. While Principles I–VIII were studied in connection with unsigned, or arithmetic numbers only, it is now very important to note that they all apply to *signed* numbers as well. The form changes described in § 37 also apply to signed numbers just the same as to arithmetic numbers.

In the statement of these principles the word *number* will from now on be understood to refer either to the ordinary numbers of arithmetic or to the signed numbers, as occasion may require. It should also be noticed that the numbers of arithmetic are used as freely in algebra as in arithmetic. It is only when we wish to distinguish them from negative numbers that they are called positive numbers.

The **number system of algebra**, as far as we have studied it, consists of the *numbers of arithmetic* together with the *negative numbers*.

INTERPRETATION AND USE OF NEGATIVE NUMBERS

69. Illustrative Problem. Divide 34 units into two parts such that one part is equal to the remainder when 3 times the other part is subtracted from 46.

Solution. Let the two numbers be represented by x and $34 - x$.

Then $$34 - x = 46 - 3x. \quad (1)$$

$S|34$, $$-x = 12 - 3x. \quad (2)$$

$A|3x$, $$3x - x = 12. \quad (3)$$

Principle II, $$2x = 12. \quad (4)$$

$D|2$, $$x = 6. \quad (5)$$

Substituting $x = 6$ in (1), $34 - 6 = 46 - 18$, or $28 = 28$.

In the above solution, equation (2) was obtained from (1) by subtracting 34 from each member. This would clearly be impossible without the use of negative numbers.

In this case the problem itself does not involve negative numbers, but in the course of its solution they naturally occur. If the negative number could not be used, we should be compelled to keep the

members of each equation positive or zero. This would be impossible, since we do not know what numbers are represented by the letters involved, and hence cannot tell by inspection whether a given term is positive or not.

70. We have seen how naturally the use of signed numbers has arisen in problems where things of opposite qualities have to be distinguished.

In solving a problem, a negative result may have a natural interpretation or it may indicate that the conditions of the problem are impossible.

A similar statement holds in reference to fractional answers in arithmetic. For example, if we say there are twice as many girls as boys in a schoolroom and 35 pupils in all, the number of boys would be $35 \div 3 = 11\frac{2}{3}$, which indicates that the conditions of the problem are impossible.

71. Illustrative Problem. The crews on three steamers together number 94 men. The second has 40 more than the first, and the third 20 more than the second. How many men in each crew?

Solution. Let $n =$ number of men in first crew.

Then $n + 40 =$ number of men in second crew,

and $n + 40 + 20 =$ number of men in third crew.

Hence $n + n + 40 + n + 40 + 20 = 94$,

and
$$3n + 100 = 94.$$
$$3n = {}^{-}6.$$
$$n = {}^{-}2.$$

Here the negative result indicates that the conditions of the problem are *impossible*.

72. Illustrative Problem. A real estate agent gained \$8400 on four transactions. On the first he gained \$6400, on the second he lost \$2100, on the third he gained \$5000. Did he lose or gain on the fourth transaction?

Solution. Since we do not know whether he gained or lost on that transaction, we represent the unknown number by n, which may be positive or negative, as will be determined by the solution of the problem.

Thus we have	$6400 + {}^{-}2100 + 5000 + n = 8400.$	(1)
Hence by IX, F,	${}^{+}9300 + n = 8400.$	(2)
By S,	$n = 8400 - 9300.$	(3)
By X,	$n = {}^{-}900.$	(4)

In this case the negative result indicates that there was a *loss* on the fourth transaction.

PROBLEMS

In the following problems give the solutions in full and state all principles used, together with the interpretation of the results:

1. A man gains \$2100 during one year. During the first three months he loses \$125 per month, then gains \$500 per month during the next five months. What is the gain or loss per month during the remaining four months?

2. A man gained \$1250 during four months. During the second month he gained \$600 more than the first month, the third month he gained \$300 less than the second, and the fourth he gained \$200 more than the third. Find the gain or loss for the first month.

3. A box containing a Christmas toy weighed 25 oz. When the toy and the packing were removed, the box weighed 20 oz. The packing weighed 7 oz. What kind of a toy was it?

4. A man rowing against a swift current rows 8 miles in 5 hours. The second hour he rows one mile less than the first, the third two miles more than the second, and the fourth and fifth one mile more each than he rowed the third hour. How many miles did he row each hour?

5. There are three trees the sum of whose heights is 108 feet. The second is 40 feet taller than the first, and the third is 30 feet taller than the second. How tall is each tree?

Find the average yearly temperature at each of the following places, the average monthly temperatures being as here given:

6. Port Conger, off the northwest coast of Greenland; $^-37°$, $^-43°$, $^-32°$, $^-15°$, $^+14°$, $^+18°$, $^+35°$, $^+34°$, $^+25°$, $^+4°$, $^-17°$, $^-30°$.

7. Franz Joseph's Land; $^-20°$, $^-20°$, $^-10°$, $0°$, $15°$, $30°$, $35°$, $30°$, $20°$, $10°$, $0°$, $^-10°$.

8. Western Baffin Land; $^-30°$, $^-30°$, $^-20°$, $0°$, $20°$, $35°$, $40°$, $35°$, $25°$, $10°$, $^-10°$, $^-20°$.

Find the average yearly loss or gain in each of the following:

9. \$1600 gain, \$8000 loss, \$24,000 gain, \$40,000 loss.

10. \$32,000 gain, \$45,000 loss, \$24,000 gain, \$42,000 loss.

11. The average yearly temperature of north central Siberia is $^-5°$. The average monthly temperatures beginning in February are: $^-50°$, $^-30°$, $0°$, $15°$, $40°$, $40°$, $35°$, $30°$, $0°$, $^-30°$, $^-50°$. Find the temperature for January.

12. The business transactions of a certain firm averaged \$1500 loss for 4 years. For the first year there was a gain of \$800, the second year a loss of \$1800, the third year a loss of \$300. What was the loss or gain for the fourth year?

13. A commercial house averaged \$15,000 gain for 6 years. What was the loss or gain the first year if the remaining years show: \$8000 gain, \$24,000 gain, \$2000 loss, \$20,000 gain, and \$50,000 gain, respectively?

14. The longitude of Boston, Massachusetts, is $^-71° \, 10'$, and that of Chicago, Illinois, is $^-87° \, 35'$. Find the longitude of Lake Chautauqua, which is midway between these.

15. The longitude of New Haven, Connecticut, is $^-72° \, 58'$, and that of Bombay, India, is $^+72° \, 48'$. The longitude of St. Paul's Cathedral, London, is midway between these. Find the longitude of the cathedral.

16. The longitude of Cincinnati, Ohio, is $^-84°\ 30'$, and that of Indianapolis, Indiana, is $^-86°\ 5'$. The longitude of Cincinnati is midway between those of Indianapolis, and Columbus, Ohio. Find the longitude of Columbus.

17. The longitude of Bristol, England, is $^-2°\ 30'$, and that of Minneapolis, Minnesota, is $^-93°\ 20'$. The longitude of Bristol is midway between those of Minneapolis and Calcutta, India. Find the longitude of Calcutta.

18. The latitude of Columbus, Ohio, is $^+40°$ and that of Winnipeg, Canada, is $^+50°$. The latitude of Columbus is midway between those of Winnipeg and Houston, Texas. Find the latitude of Houston.

19. The longitude of Montreal, Canada, is $^-73°\ 40'$ and that of Baltimore, Maryland, $^-76°\ 40'$. Find the longitude of Philadelphia, which is midway between these.

20. The latitude of Lima, Peru, is $^-12°$ and that of Buenos Ayres, Argentina, $^-34°\ 35'$. The latitude of Lima is midway between those of Buenos Ayres and Caracas, Venezuela. Find the latitude of Caracas.

21. The longitude of Providence, Rhode Island, is $^-71°\ 25'$ and that of Fargo, North Dakota, $^-96°\ 50'$. The longitude of Fargo is midway between those of Providence and Seattle, Washington. Find the longitude of Seattle.

CHAPTER III

INVOLVED NUMBER EXPRESSIONS

73. Double Use of the Signs $+$ and $-$. In the preceding chapter it has been found that the negative quality may be regarded as implying subtraction and the positive quality as implying addition. It was for this reason that $^+$ and $^-$ were selected as symbols for the words "positive" and "negative."

74. It is now possible to dispense with these special signs of quality. For, by Principle X, $a - {}^+b$ and $a + {}^-b$ are the same in effect, and likewise $a - {}^-b$ and $a + {}^+b$. Hence, omitting the positive signs (§ 45), we may write

$$a + {}^-b = a - b \text{ and } a - {}^-b = a + b.$$

One set of signs is, therefore, sufficient as symbols both of operation and of quality.

E.g. $5 - 7$ means either $5 + {}^-7$ or $5 - {}^+7$, and in either case equals ${}^-2$, which we now write -2. Thus $5 - 7$ equals -2, which is read, 5 *minus* 7 *equals negative* 2.

Ex. 1. ${}^-5 \cdot {}^-4 = {}^+20$ is now written $-5 \cdot -4$, or $(-5)(-4)$, $= +20$,

and ${}^-5 \cdot {}^+4 = {}^-20$ is written $-5 \cdot +4$ or, $(-5)(+4), = -20$.

Ex. 2. $5(9a + {}^-2a) = 5(9a - 2a)$.
By IV, $= 45a - 10a$.
By II, $= 35a$.

Ex. 3. $5a + {}^-4({}^-3a - {}^+7b) = 5a - 4(-3a - 7b)$.
By IV, XI, $= 5a + 12a + 28b$.
By I, $= 17a + 28b$.

EXERCISES

Rewrite the following, using one set of signs, and then perform the indicated operations.

1. $7 - {}^{+}3 + {}^{-}8$.
2. ${}^{-}9 - {}^{-}3 + {}^{-}12$.
3. ${}^{-}4\,a + {}^{-}5\,a + {}^{-}6\,a$.
4. ${}^{-}3 \cdot 5\,x + 4 \cdot 7\,x - 8 \cdot 8\,x$.
5. $27\,abc + {}^{-}35\,abc - 2\,abc$.
6. $5({}^{-}2 + {}^{-}3) + 4(5 - {}^{-}7)$.
7. ${}^{-}3(7-2) - 8(6 + {}^{-}9)$.
8. $3(4\,a - {}^{-}5\,b) - 11({}^{-}2\,a + {}^{-}3\,b)$

Rewrite the following expressions, using special signs of quality so that all signs of operation shall indicate addition:

9. $5 - 8 - 14 = 5 + {}^{-}8 + {}^{-}14$.
10. $-7 + 8 - 18$.
11. $-4\,a + 5\,a - 17\,a$.
12. $-7 \cdot 4\,x + 7 \cdot 4\,x - 8 \cdot 4\,x$.
13. $56\,ay - 72\,ay + 7ay$.
14. $3(2-5) + 5(3-7)$.
15. $-3(8-6) - 4(6-9)$.
16. $8(4\,t - 5\,n) - 5(-t + 4\,n)$.

POLYNOMIALS

75. We have found that the solution of problems leads us to build **involved number expressions** out of single number symbols.

E.g. If x is a number representing my age in years, then $2(x-10)$ is double the number representing my age 10 years ago, and $2\,[(x-10)+(x+15)]$ is the number representing twice the sum of my ages 10 years ago and 15 years hence.

Number expressions are now to be studied more in detail.

76. Definition. A number expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial**. Each of the parts thus connected together with the sign preceding it is called a **term**.

E.g. $5\,a - 3\,xy - \frac{2}{3}\,rt + 99$ is a polynomial whose terms are $5\,a$, $-3\,xy$, $-\frac{2}{3}\,rt$, and $+99$. The sign $+$ is understood before $5\,a$.

77. **Definitions.** A polynomial of two terms is called a **binomial**, one of three terms is called a **trinomial.** A term taken by itself is called a **monomial.**

E.g. $5a-3xy$ is a binomial; $5a-3xy-\frac{2}{3}rt$ is a trinomial; $5a$, $-3xy$, $-\frac{2}{3}rt$ are monomials.

According to the above definition $x+(b+c)$ may be called a binomial notwithstanding it is equivalent to the trinomial $x+b+c$.

In this case x is called a **simple** term and $(b+c)$ a **compound** term. Likewise we may call $3t+4x-5(a+b)y$ a trinomial having the simple terms $3t$, $+4x$, and the compound term $-5(a+b)y$.

It should be clearly understood that a negative or positive sign before a compound term (as well as before a simple term) applies to the number represented by the whole term.

78. **Definition.** Two terms which have a factor in common are called **similar with respect to that factor.**

E.g. $5a$ and $-3a$ are similar with respect to a; $-3xy$ and $-7x$ are similar with respect to x; $5a$ and $-5b$ are similar with respect to 5; $7abc$ and $-\frac{2}{3}abc$ are similar with respect to abc.

Similar terms may be combined by Principles I, II, and IX.

E.g. $5a-3a=(5-3)a=2a$; $-3xy-7x=-x(3y+7)$; $5a-5b=5(a-b)$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

79. In adding or subtracting polynomials the work may be conveniently arranged by placing the terms in columns, each column consisting of terms which are similar.

Ex. Add $5x-6y+4z+5at$, $-3x+11y-16z-9bt$, and $-7y+8z$.

Arranging as suggested and applying Principles I, II, and IX, we have

$$\begin{array}{r}
5x - 6y + 4z + 5at \\
-3x + 11y - 16z - 9bt \\
- 7y + 8z \\
\hline
2x - 2y - 4z + t(5a - 9b)
\end{array}$$

$5x$ and $-3x$ are similar with respect to their common factor x. Hence by Principle I we add the other factors 5 and -3, obtaining $(5-3)x = 2x$.

Likewise we add $+5at$ and $-9bt$ with respect to the common factor t, obtaining $(5a-9b)t$. In the second column the sum is $(-6+11-7)y = -2y$, and in the third column the sum is $(+4-16+8)z = -4z$.

Check by giving convenient values to x, y, z, t, a, and b.

EXERCISES

1. Add $7b - 3c + 2d$; $-2b + 8c - 13d$.

2. Add $6x - 3y + 4t - 7z$; $x - 5y - 3t$; $4x - 4y + 8t$.

3. Add $5ac + 3bc - 4c + 8b$; $2b + 3c - 2bc - 3ac$; $4b + 4c + bc - ac$; $2bc + 4ac + c$; $3b - 4c$.

4. Add $3 \cdot 4 \cdot 7 - 5x + 5abx$; $3aby + 3x - 5 \cdot 4 \cdot 7$; $7x + 2 \cdot 4 \cdot 7$; $5aby - 3x - 5 \cdot 4 \cdot 7$; $45x + abx - 4 \cdot 7$.

5. Add $3(x-5) + a(c+b) + b(x-y)$; $b(c+b) - a(x-y) + 8(x-5)$; $7(c+b) - 4(x-y)$; $3(x-y) + (x-5)$.

6. Add $16(a+b-c) - 3(x-y) + 2(a-b)$; $2(x-y) - 3(a-b) + (a+b-c)$; $7(a-b) + a(x-y) - b(a+b-c)$.

7. Add $a(a-b) - c(x+y) + d(x-z) - 4abc$; $c(x-z) - d(x+y) + (a-b) + 2abc$; $c(a-b) + mabc + 3(x-z) + 8(x+y)$.

8. Add $7a - 4x + 12z$; $ba - 3x + cz$; $2ba + 4x - 3cz$.

9. Add $(a-b)-3(c-d)+m(a+d)$; $c(a-b)+a(c-d)$.

10. Add $34ax+4by-3z$; $2by+5z$; $3bx-7y+5dz$.

11. Add $3b+4cd-2ae$; $ab-3cd+3ae$; $3cd-2ab$.

12. Add $7ax-13by+5$; $9ax+8by-4$; $3b-12ax$.

13. Add $5(a+b)-3(c-d)$; $3(c-d)-8(a+b)$; $-2(a+b)$.

14. Add $3+4(c-d)-5(a-b-c)$; $4(a-b-c)+5(c-d)$; $3(a-b-c)-9(c-d)+12$.

15. Add $11(c-9)+3(x+y)+21wu$; $-71wu-5(x+y)-13(c-9)$.

16. Add $5ab-3\cdot 7\cdot 9+5(x-1)$; $5\cdot 9\cdot 7+3ab-2(x-1)$; $3(x-1)-4\cdot 7\cdot 9+2ab$.

17. Add $31\cdot 50-43\cdot 74+2\cdot 18$; $21\cdot 74+7\cdot 18-56\cdot 50$; $-12\cdot 18+42\cdot 50-6\cdot 74$.

18. Add $7(x-y)-4(x+y)+4\cdot 7$; $9(x+y)+3(x-y)-9\cdot 7$; $6(x-y)+2\cdot 7-3(x+y)$.

19. Add $16xy-13\cdot 64$; $15ab-2xy$; $34\cdot 64-3xy+2ab$; $14\cdot 64-3xy-2ab$.

80. The subtraction of polynomials is illustrated by the following example:

From $15ab-17xy+11rt$ subtract $-5ab+4xy-5nt$.

Arranging as on page 75 and applying Principles II and X,

$$\begin{array}{r} 15ab-17xy+11rt \\ -5ab+\ \ 4xy-\ \ 5nt \\ \hline 20ab-21xy+t(11r+5n) \end{array}$$

As suggested in § 60, it is sufficient to change the signs of the subtrahend *mentally*, rather than to rewrite them before adding to the minuend.

EXERCISES

1. From $9x + 3y - 11z$ subtract $-5x + 8y - 3z$.

2. From $12ab - 3cd + 12xy$ subtract $3ab + 2cd - 11xy$.

3. From $9xc + 4ad - 3cz + 5y$ subtract $3y - 3ad + 5cz$.

4. From $13t + 5mx - 5cv$ subtract $2t - 4mx - 3cv$.

5. From $3v - 2w + 5mn - 4xz$ subtract $-v + 5w - 3mn$.

6. From $31b + 4xy + 16ax - 4$ subtract $8b - 5xy - 3ax$.

7. From $4 - 3a - 5xz - 3vy - x$ subtract $7a + 2xz + 4vy$.

8. From $8xy - 3x + 4y$ subtract $-2xy + 13w + 4x - 2y$.

9. From $2ab - 5 + 7v + 13abc$ subtract $3ab + v + 8abc$.

10. From $8cxa - 4yb - 3yc$ subtract $4bxa + 2yb + 4yc - 49$.

11. From $31 \cdot 45 - 7xy$ subtract $12 \cdot 45 + 9xy$.

12. From $3abc - 4 \cdot 28 + 2(x + y) - 3xy$ subtract $6 \cdot 28 + 4xy - 3(x + y) + 8abc$.

13. From $7 \cdot 3 \cdot 5 + 9(xy - z) + 4 \cdot 3(a + b)$ subtract $8(xy - z) - 8 \cdot 3(a + b) + 8 \cdot 3 \cdot 5$.

14. From $5ax - 3by + 4ax + 5by$ subtract $5by + 3ax + 7by$.

15. From $19(r - 5s) + 13(5x - 4) + 7(x - y)$ subtract $17(5x - 4) - 5(x - y) - 11(r - 5s)$.

16. From $16 - 15 \cdot 30 + 14(x - 5yz) - 13(5y - z)$ subtract $32 - 16 \cdot 30 + 8(5y - z)$.

17. From $-41 \cdot 3 + 13 \cdot 4 \cdot 16$ subtract $7 \cdot 4 \cdot 16 - 8 \cdot 3$.

18. From $a(b + c) + 4(m + n) - 16c$ subtract $9(m + n) + 31c - d(b + c)$.

19. From $5(7x - 4) + 3(5y - 3x) + 5 \cdot 7$ subtract $8 \cdot 7 - 9(7x - 4) + 8(5y - 3x)$.

20. From $15 \cdot 48 + 8ab + 49x$ subtract $7 \cdot 48 - 9ab - 14x$.

EXERCISES IN ADDITION AND SUBTRACTION

1. Add $5x-3y-7r+8t$, $-7x+18y-4r-7t$, $-20x-24y+18r-15t$, and $13x+15y+11r+6t$.

Check the sum by substituting $x=1$, $y=1$, $r=1$, $t=1$.

2. Add $17a-9b$, $3c+14a$, $b-3a$, $a-17c$, and $a-3b+4c$. Check for $a=1$, $b=2$, $c=3$.

3. Add $2x+3y-t$, $-6y+8t$, $-x+y-t$, $-4t+7x$, and $3y$. Check for $x=2$, $y=3$, $t=1$.

4. Add $17r+4s-t$, $2t+3u$, $2r-3s+4t$, $5u-6t$, $7r-3s+8u$, and $8r-2t+6u$. Check by putting each letter equal to 1; also equal to 2.

5. Add $3h+2t+4u$ and $h+3t+3u$. Check by putting $h=100$, $t=10$, $u=1$; *i.e.* $324+133=457$.

6. Add $4h+3t+u$ and $3h+2t+7u$. Check as in 5.

7. Write 247, 323, 647, 239, and 41, as number expressions like those in 5 and 6 and then add them.

8. Add $4t-u$, $5t-u$, $6t-u$, $7t-u$, and $8t-u$. Check for $t=10$, $u=1$; also $t=1$, $u=1$.

9. Add 647, 391, 276, and 444 as in example 7.

10. Simplify: $3xyz-2xyz+5xyz-4xyz+xyz-xyz$.

11. Subtract $5a-3b+6c$ from $-8a+7b-11c$ and check.

12. From $7xy+8xz+9yz$ take $17xy-19xz-20yz$.

13. From $6x-3y$ take $8y-3z$.

14. From $3p-4q+8r$ take $7p-11r+11q$.

15. From $a+b+c$ take $x-y+z$. *Suggestion*: By Principle VII, $a+b+c-(x-y+z)=a+b+c-x+y-z$.

16. From $2x-3y$ take $5x+7y+2a-3b$.

17. From the sum of $18abc-27xyz+13rst$ and $-11abc+16xyz-52rst$ take $67rst-39abc$.

18. To the difference between the subtrahend $15x-18y+27z$ and the minuend $117x+97y-81z$ add $4x-6y+3z$.

19. Add $11(x-y)+15(a-b)$ and $-20(x-y)-37(a-b)$ and from the sum subtract $135(x-y)-213(a-b)$.

20. Add $6ax+7bx-8cx$, $-11ax-18bx+25cx$, and $19ax-16cx+24bx$.

21. From $13mn-25mp+36mq$ subtract $18mn+23mp$.

22. Add by Principle I, $6\cdot 3\cdot 9-11\cdot 5\cdot 7+16\cdot 9\cdot 11$ and $-8\cdot 3\cdot 9+24\cdot 5\cdot 7-23\cdot 9\cdot 11$.

23. From $83\cdot 9+78\cdot 13$ subtract $57\cdot 9-93\cdot 13+85\cdot 17$.

24. From $3h+4t+2u$ subtract $h+5t+3u$. Check.

25. Subtract $7(a-x)-10(b-y)$ from $13(a-x)+5(b-y)$.

81. In solving problems it is often unnecessary to arrange the work of addition and subtraction in columns as above. In most cases the operations can be readily indicated by means of parentheses as illustrated in the following example:

From the sum of $3a+4b$ and $5a-8b$ subtract $7a-6b$.

Indicate these operations thus: $3a+4b+(5a-8b)-(7a-6b)$.
Applying Principle VII, we have $3a+4b+5a-8b-7a+6b$.
Collecting similar terms, $3a+5a-7a+4b-8b+6b$.
Finally, applying Principles I and II, we obtain $a+2b$.
After a little practice the last two steps can be taken at once.

EXERCISES AND PROBLEMS

Perform the following indicated operations by collecting similar terms at once without arranging in columns:

1. $15+(7-9x)-(-7x-9)+9$.
2. $7+5y-(3y+2)+(8-4y)$.
3. $2a+3+(4a-5)-(11a-14)$.
4. $32b-(17b-12)-(4b-13)$.

5. $16c-(41-7c)+(15-8c)$.

6. $-(5a-3c)-(2c-8a)+3a$.

7. $-(-12x-7y-15x)-(-9y+8x+3y)$.

8. $(19x+4y-32x-17x)-12x-(49y+18x-70x)$.

9. $17a-3-(7a-2)+(6a-5)$.

10. $5x-(8-4x+7y)+(5x+3)-(5y+3x-99)$.

11. $-(3a+5b-7c)+(8a-4c)-(9c-4b+4a)-91a$.

12. $7-(4-4c+2d-2a)+31c-(4-2a-5d)-(-8c)$.

13. $(41ab-21c+4)-(36c+15-78ab)+(13c-90ab-8)$.

14. $9by-(4c-8by-13)-2c-16-(34by-12c+8by)$.

15. $6mn+(-9m-7n+14)-8n+(13mn-17m)+34mn$.

16. $34ax-(-17ax+42)+8x-(14a+24ax-7)$.

17. $19-(+2-7a-4b+11ab)-(-2b+8ab+4a)$.

18. $41by-(4b-13y+17by)-(-5b-17by+13y)$.

19. $39rs-20s-19r-(7rs+8s-19r)-(15r-5s-56)$.

20. $a(3x-2y-z)-(5ax-ay+3z)+az$.

21. $5(4h+3bk-7br)-b(15k-35r)-20h$.

22. The altitude of Popocatepetl is 1716 feet less than that of Mt. Logan, and the altitude of Mt. St. Elias is 316 feet greater than that of Popocatepetl. Find the altitude of each mountain, the sum of their altitudes being 55,384 feet.

23. The Ganges River is 1800 miles shorter than the Amazon, and the Orinoco is 300 miles shorter than the Ganges. The sum of their lengths is 6900 miles. How long is each?

24. A cubic foot of red oak weighs 35 pounds less than 2 cubic feet of cherry wood, and 21 pounds more than a cubic foot of chestnut; while a cubic foot of chestnut weighs 100 pounds less than 3 cubic feet of cherry. Find the weight of each kind of wood per cubic foot.

25. Lead weighs 259 pounds more per cubic foot than cast iron, and 166 pounds more than bronze; while a cubic foot of bronze weighs 807 pounds less than 3 cubic feet of iron. Find the weight per cubic foot of each metal.

26. Green glass weighs 60 pounds per cubic foot less than dense flint glass, and 8 pounds more than crown glass; while a cubic foot of crown glass weighs 293 pounds less than 2 cubic feet of flint glass. Find the weight per cubic foot of each.

27. Europe has 12 million inhabitants less than 10 times as many as South America, and North America has 29 million more than twice as many as South America. If 3 times the population of North America be subtracted from that of Europe, the remainder is 65 million. How many inhabitants has each continent?

28. The length of the Rio Grande River is $\frac{3}{4}$ that of the Volga, and the Mississippi is 600 miles less than twice as long as the Volga. If $\frac{1}{3}$ the length of the Mississippi be subtracted from that of the Rio Grande, the remainder is 400 miles. Find the length of each river.

29. In 1900 the total wealth of the United States was 1532 million dollars more than 13 times as great as in 1850, and 9016 million more than twice as great as in 1880. The total wealth in 1880 was 174 million less than 6 times as great as in 1850. What was the wealth in each of the three years?

30. The money circulation of the United States in 1880 was 13 million dollars more than 60 times that in 1800 and in 1905 it was 188 million more than 150 times that in 1800. One-seventh of the amount in 1880 plus $\frac{1}{4}$ the amount in 1900 was 786 million. Find the circulation for each year.

31. The total bank deposits in the United States in 1905 were 3127 million dollars less than twice as great as in 1900 and 681 million more than 5 times as great as in 1880. The deposits in 1880 were 5105 million less than in 1900. Find the deposits for each of the three years.

32. The amount of deposits in savings banks in the United States in 1905 was 703 million dollars greater than in 1900, and 183 million less than 4 times that in 1880. The amount in 1900 was 67 million less than 3 times as great as in 1880. Find the deposits for each of the three years.

33. The total value of the farms in the United States in 1880 was 280 million dollars more than 3 times their value in 1850, and 8333 million less than their value in 1900. The value of the farms in 1880 was 1924 million less than $\frac{1}{2}$ their value in 1900. Find the value in each of the three years.

34. The value of the manufactures in the United States in 1900 was 811 million dollars more than 12 times that in 1850, and 3071 million less than 3 times that of 1880; while the value in 1880 was 744 million less than 6 times that in 1850. Find the value of the manufactures for each year.

MULTIPLICATION OF POLYNOMIALS

82. Illustrative Problem. A rectangular field is 12 rods longer than it is wide; a second rectangular field which is 4 rods shorter and 2 rods wider than the first has an area of 80 square rods less. What are dimensions of the first field?

Solution. Let $w =$ number of rods in the width of first field.

Then $w + 12 =$ number of rods in the length of first field,

and $w(w + 12) =$ width $\times$ length, or area of first field;

also $w + 2 =$ width of second field,

and $w + 8 =$ length of second field.

Hence $(w + 2)(w + 8) =$ width $\times$ length, or area of second field.

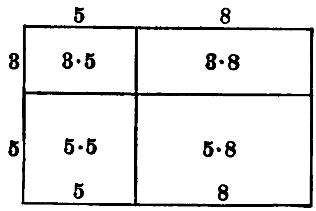
But the area of the second field is 80 square rods less than that of the first.

Hence $$(w + 2)(w + 8) = w(w + 12) - 80. \qquad (1)$$

To solve this equation it is necessary to obtain the product of the two binominals $w + 2$ and $w + 8$ without first combining the terms of each binominal. In order to determine how these

are to be multiplied, let us consider two binomials in each of which the terms can be combined if desired.

E.g. Consider the product of the binomials $5 + 3$ and $5 + 8$. This may be represented by the area of a rectangle 8 feet wide and 13 feet long. That is, $(5+3)(5+8) = 8 \cdot 13 = 104$ square feet. But such a rectangle may be divided into four rectangles, as in the figure.



Hence the area may be expressed as the sum of four areas. Thus, $(5 + 3)(5 + 8) = 5 \cdot 5 + 5 \cdot 8 + 3 \cdot 5 + 3 \cdot 8 = 104$, as before.

83. The product $5 \cdot 5$ is abbreviated to 5^2, the small figure indicating that 5 is to be used as a factor twice. It is read *the square of* 5, *or* 5 *squared*, since it represents the area of a square whose sides are each 5.

The second method here used for multiplying $(5+3)(5+8)$ is applicable to any similar case, and does not depend upon the possibility of first combining the terms of the binomials.

Applying this method to the binomials in equation (1) of the problem in § 82, we have

$$w^2 + 8w + 2w + 16 = w^2 + 12w - 80. \quad (2)$$

Subtracting w^2 from both sides and applying Principle I,

$$10w + 16 = 12w - 80. \quad (3)$$

Subtracting $10w$ from both sides and adding 80 to both sides,

$$96 = 2w. \quad (4)$$

Hence by D, $\quad 48 = w$, the width of the field;

and $\quad 60 = w + 12$, the length of the field.

Check by substituting $w = 48$ in equation (1), and also by showing that the numbers 48 and 60 satisfy the conditions stated in the problem.

84. In a manner similar to that just illustrated we may multiply two trinomials.

E.g. The product of $a + b + c$ and $m + n + r$, in which the letters represent any positive numbers, may represent the area of a rectangle, divided into small rectangles as follows:

	m	n	r
a	am	an	ar
b	bm	bn	br
c	cm	cn	cr

Hence, the product is:

$(a + b + c)(m + n + r) = am + bm + cm + an + bn + cn + ar + br + cr,$ in which each term of one trinomial is multiplied by every term of the other, and the products are added.

Evidently the same process is applicable to the product of two such polynomials each containing any number of terms.

EXERCISES AND PROBLEMS

Find each of the following products in two ways:

1. $(3 + 7 + 10)(2 + 6)$.
2. $(5 + 11)(13 + 10 + 5)$.
3. $(6 + 11)(6 + 7)$.
4. $(15 + 8)(15 + 4)$.
5. $(7 + 13)(a + b)$.
6. $(m + n)(11 + 5 + 4)$.
7. $42 \cdot 36 = (40 + 2)(30 + 6)$.
8. $28 \cdot 73 = (20 + 8)(70 + 3)$.

Find as many as possible of the following indicated products in two ways:

9. $(a + b)(c + d)$.
10. $(x + 4)(x + 3)$.
11. $(x + y + z)(a + b + c)$.
12. $(11 + 13)(r + t)$.
13. $(5 + x)(5 + 7)$.
14. $(3 + 8)(2 + 4 + 6)$.
15. $(x + 7)(3x + 4)$.
16. $(a + 4)(3a + 1)$.
17. $(3 + x)(2 + 5x)$.
18. $(a + b)(3a + 7b)$.
19. $(x + y)(2x + 3y)$.
20. $(7x + 4x)(x + 8)$.

21. A rectangle is 7 feet longer than it is wide. If its length is increased by 3 feet and its width increased by 2 feet, its area is increased by 60 square feet. What are its dimensions?

22. A field is 10 rods longer than it is wide. If its length is increased by 10 rods and its width increased by 5 rods, the area is increased by 640 square rods. What are the dimensions of the field?

23. A farmer has a plan for a granary which is to be 12 feet longer than wide. He finds that if the length is increased 8 feet and the width increased 2 feet, the floor space will be increased by 160 square feet. What are the dimensions?

24. If the length of a rectangular flower bed is increased 3 feet and its width increased 1 foot, its area will be increased by 19 square feet. What are its present dimensions, if its length is 4 feet greater than its width?

85. Polynomials with Negative Terms. The polynomials multiplied in the foregoing exercises contain positive terms only. The same process is applicable to polynomials containing negative terms, as is seen in the following examples:

Ex. 1. Find the product of $(7-4)$ and $(3+5)$. This product, written out term by term, would give

$$\begin{aligned}[7+(-4)](3+5) &= 7\cdot 3+7\cdot 5+(-4)\cdot 3+(-4\cdot 5)\\ &= 21+35-12-20=24.\end{aligned}$$

Also $(7-4)(3+5)=3\cdot 8=24.$

Ex. 2. Multiply $7-4$ and $8-3$.

$$\begin{aligned}(7-4)(8-3) &= [7+(-4)][8+(-3)]\\ &= 7\cdot 8+7\cdot(-3)+(-4)\cdot 8+(-4)(-3)\\ &= 56-21-32+12=15.\end{aligned}$$

Also $(7-4)(8-3)=3\cdot 5=15.$

EXERCISES

Find each of the following products in two ways, as in the above examples:

1. $(11-7)(6+5)$.
2. $(22-13)(3+7)$.
3. $(8-5)(7-3)$.
4. $(17-9)(29-4)$.
5. $(2-3)(4b-7b)$.
6. $(3+x)(7x-3x)$.
7. $(8-3)(8-2)$.
8. $(9-13)(9-17)$.

Perform the following indicated operations:

9. $(a-b)(c+d)$.
10. $(a-b)(c-d)$.
11. $(x-4)(x-5)$.
12. $(a+b-c)(m-n)$.
13. $(a-b)(7a+3b)$.
14. $(5-y)(5x+3y)$.
15. $(2a-3b+c)(m+n)$.
16. $(v-t)(7v-5t)$.

86. The preceding exercises illustrate the following principle:

Principle XIII. *The product of two polynomials is found by multiplying each term of one by every term of the other, and adding these products.*

In case some of the partial products are negative, these are combined by Principle IX.

If there are similar terms in either polynomial, these should usually be added first, thus putting each polynomial in as simple form as possible.

E.g. $(3x+2-2x)(4x+3-3x)=(x+2)(x+3)$
$$=x^2+2x+3x+6=x^2+5x+6.$$

EXERCISES AND PROBLEMS

Perform the following indicated operations:

1. $(x-7)(3x-4+8)$.
2. $(1-2x+x-3)(2x+4a+7x)$.
3. $(4a-x-3a)(2x+4a+7x)$.

4. $(5x+3y-4x-2y)(6y+3x-2y+y)$.
5. $(13a-b-12a)(2b-3a)$.
6. $(xy-5xy+4y)(8y-3-7y)$.
7. $(11ab+3a)(2b-3b+5)$.
8. $(6-4x+3x)(7x+y-3x+1)$.
9. $(13x-12x-y+3)(5x-3y+xy+5)$.
10. $(37-13n+a)(a-n+8)$.
11. $(x-2+y)(4y-3x)$.
12. $(11b-a-10b)(6a-3b-2a)$.
13. $(7+y-x)(2y+x-1)$.
14. $(5x+3y-1)(x-2)$.
15. $(-8a-1+7a)(5a-8-3a)$.

Solve the following equations, in each case verifying the solution by substituting in the given equation the result found for the unknown number.

16. $(x+2)(x+3)=(x-3)(x+10)+10$.
17. $(5x-4)(6-x)-97=(x-1)(6-5x)$.
18. $(3n-1)(18-n)=(n+6)(16-3n)$.
19. $(7-a)(9a-8)=31+(36-9a)(a+2)$.
20. $(4a+4)(a-3)=(4a+1)(a+7)-13a+221$.
21. $(n+6)(3n-4)-14=(n+8)(3n-3)$.
22. $(8n+6)(10-n)+150=(1-n)(8n+3)$.
23. $(a-1)(13-6a)=(6a-3)(8-a)-21$.
24. $(7x-13)(6-x)-(x+4)(3-7x)=70$.

25. A club makes an equal assessment on its members each year to raise a certain fixed sum. One year each member pays a number of dollars equal to the number of members of the club less 175. The following year, when the club has 50 more members, each member pays \$5 less than the preceding year. What was the membership of the club the first year and how much did each pay?

26. There are two numbers whose difference is 6 and whose product is 180 greater than the square of the smaller. What are the numbers?

27. There are four consecutive even integers such that the product of the first and second is 40 less than the product of the third and fourth. What are the numbers?

28. There are four consecutive integers such that the product of the first and third is 223 less than the product of the second and fourth. What are the numbers?

29. There are four numbers such that the second is 5 greater than the first, the third 5 greater than the second, and the fourth 5 greater than the third. The product of the first and second is 250 less than the product of the third and fourth. What are the numbers?

30. Prove that for any four consecutive integers the product of the first and fourth is 2 less than the product of the second and third.

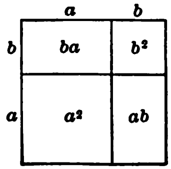
SQUARES OF BINOMIALS

87. Just as x^2 is written instead of $x \cdot x$, so $(a+b)^2$ is written instead of $(a+b)(a+b)$. The square of a binomial is found by multiplying the binomial by itself as in § 86.

E.g. $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2$.

Hence $$(a+b)^2 \equiv a^2 + 2ab + b^2.$$

This product is illustrated in the accompanying figure, and is evidently a special case of the type exhibited in the figure, page 83.



Translated into words this identity is: *The square of the sum of any two numbers is equal to the square of the first plus twice the product of the two numbers plus the square of the second.*

88. Similarly we obtain the square of the difference of two numbers.

$$(a-b)^2 \equiv a^2 - 2\,ab + b^2.$$

Translate this identity into words.

While these squares are ordinary products of binomials and can always be obtained according to Principle XIII, they are of special importance and should be studied until they can be reproduced from memory at any time.

EXERCISES AND PROBLEMS

Perform the following indicated operations:

1. $(a+b)^2$.	4. $(a-2)^2$.	7. $(4+9)^2$.
2. $(17-3)^2$.	5. $(21-b)^2$.	8. $(c-\frac{3}{7})^2$.
3. $(6+a)^2$.	6. $(x-7)^2$.	9. $(x-\frac{1}{3})^2$.

10. $(r-s)^2-(r+s)^2+(r-s)(r+s)$.

Check each of the above by substituting special values for the letters and combining the terms of the binomial before squaring.

11. Find the square of 42 by writing it as a binomial, $40+2$.

12. Square the following numbers by writing each as a binomial sum: 51, 53, 93, 91, 102, 202, 301.

13. Find the square of 29 by writing it as a binomial, $30-1$.

14. Square the following numbers by first writing each as a binomial difference: 28, 38, 89, 77, 99, 198, 499, 998, 999.

Solve the following equations, verifying each solution:

15. $(a+4)^2+(a-1)(2\,a+5)=(a+4)(3\,a+2)$.

16. $(a-1)(3\,a-1)-(a+1)^2=2\,a^2-18$.

17. $(6-a)^2+(a-3)(2\,a-5)=(3\,a+1)(a-3)+84$.

18. $(7\,a-18)(a+4)-(a-1)^2=6\,(a+2)^2-79$.

19. $(2b-30)(b-1)-5b^2=6b-3(b+5)^2+65.$

20. $(5-b)(6b+5)+4(b-3)^2=20-2(b+1)^2+3+16b.$

21. $(5-c)^2+(7-c)^2+(9-c)^2=(c-1)(3c-58)-93.$

22. $(5c-3)(2+c)-4(c-1)^2=(c+1)^2+54.$

23. $(8-4c)(5-c)=(c+1)^2+(c+3)(3c-8)+218.$

24. $(4y-9)(y-5)-5(y-4)^2=(8-y)(4+y)-82.$

25. $(y+6)^2-3(y-1)^2+(4-y)(5-2y)=25y+9.$

26. $(y-1)^2+4(y+1)^2+(1-y)(5y+6)=15y-29.$

27. $x(x+3)+(x+1)(x+2)=2x(x+5)+2.$

28. $x^2=(x-3)(x+6)-12.$

29. $(5+5x)(3-x)+2(x+1)^2+3(x+1)(x-7)=17(x+1).$

30. $(8+3x)(4-x)+(x-1)(x-2)+2(x+5)^2=105.$

31. There is a square field such that if its dimensions are increased by 5 rods its area is increased 625 square rods. How large is the field?

Suggestion: If a side of the original field is w, then its area is w^2, and the area of the enlarged field is $(w+5)^2$.

32. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

33. A boy has a certain number of pennies which he attempts to arrange in a solid square. With a certain number on each side of the square he has 10 left over. Making each side of the square 1 larger, he lacks 7 of completing it. How many pennies has he?

34. A room is 7 feet longer than it is wide. A square room whose side is 3 feet greater than the width of the first room is equal to it in area. What are the dimensions of the first room?

35. Find two consecutive integers whose squares differ by 51.

36. Find two consecutive integers whose squares differ by 97.

37. Find two consecutive integers whose squares differ by a. Show from the form of the equation obtained that a must be an *odd* integer.

38. There are four consecutive integers such that the sum of the squares of the last two exceeds the sum of the squares of the first two by 20. What are the numbers?

39. Two square pieces of land require together 360 rods of fence? If the difference in the area of the pieces is 900 square rods, how large is each piece? (*Hint:* $x^2-(90-x)^2=900$.)

40. There is a square such that if one side is increased by 12 feet and the other side decreased by 8 feet the resulting rectangle will have the same area as the square. Find the side of the square.

41. A regiment was drawn up in a solid square. After 50 men had been removed the officer attempted to draw up the square by putting one man less on each side, when he found he had 9 men left over. How many men in the regiment?

42. There is a rectangle whose length exceeds its width by 11 rods. A square whose side is 5 rods greater than the width of the rectangle is equal to it in area. What are the dimensions of the rectangle?

89. Thus far the processes of algebra have all been based upon thirteen **fundamental principles**, together with certain obvious *form changes* indicated in § 37. These latter are of general use in elementary arithmetic and need no special emphasis here. The principles, however, for the most part refer to methods not common in arithmetic. These should now be carefully reviewed and a list of them made for convenient reference.

REVIEW QUESTIONS

1. How would $3 \cdot 5$ and $7 \cdot 5$ be added in arithmetic? Why cannot $3n$ and $7n$ be added in the same manner? State in full the principle by which $3n$ and $7n$ are added. In this example what number is represented by n? Test the identity $3n + 7n \equiv 10n$ by substituting any convenient value for n.

2. How is $5 \cdot 9$ subtracted from $11 \cdot 9$ in arithmetic? In what different manner may this operation be performed? Why is it sometimes necessary to perform subtraction in the second way? State in full the principle by which $12x$ is subtracted from $31x$. In the identity $31x - 12x \equiv 19x$, what number is represented by x? Test the equality by substituting any convenient number for x.

3. How is the product $2 \cdot 3 \cdot 5$ multiplied by 4 in arithmetic? In what different way may this multiplication be performed? Why should it ever be performed in the second way? State in full the principle by which $2ax$ is multiplied by 3.

4. How is $11 + 3$ multiplied by 4 in arithmetic? In what different way may this operation be performed? Why is it sometimes necessary to multiply in the second way? State in full the principle by which $a + 8$ is multiplied by 7.

What principles are used in performing the following indicated operations?

$$ax + bx + cx, \qquad aby + cby, \qquad c(4a + 2b),$$
$$3y - by + cy, \qquad 41a - 32b - 17a + 80b,$$
$$5(16a - 3b), \qquad -3(6x - 7y), \qquad a(11 - b).$$

5. Divide $2 \cdot 4 \cdot 6 \cdot 20$ by 2 without first performing the multiplication indicated in $2 \cdot 4 \cdot 6 \cdot 20$. Do this in several ways and show that all the quotients obtained are equal. State in full the principle used.

What principles are used in the following operations?

$$3 \cdot 5\,ab = 15\,ab, \qquad c \cdot 5\,b = 5\,(bc), \qquad 20\,ab \div 4 = 5\,ab,$$
$$16\,xy \div x = 16\,y, \qquad 3t + 15t = 18t, \qquad 78h - 41h = 37h.$$

6. How is $12+18$ divided by 6 in arithmetic? In what different way may this division be performed? Why is it sometimes necessary to perform division in the second way? State in full the principle used in performing the operation $(6x+9y)\div 3$. What principle is used in performing each of the following indicated operations?

$$5(a+x), \qquad 3y-4y, \qquad (24x+9y)\div 3, \qquad .5x+ax.$$

7. Define equality; equation; identity. State in detail how the equation and the identity differ. Give an example of each.

8. In what ways may an equation be changed into another equation such that any number which satisfies either also satisfies the other?

Describe some of the operations which change the *form* of the members of an equation, but not their *value*.

State Principle VIII in full.

9. Name several pairs of opposite qualities all of which are conveniently described by the words "positive" and "negative." What symbols are used to replace these words when applied to numbers?

10. When loss is added to profit, is the profit increased or decreased? What algebraic symbols are used to distinguish the numbers representing profit and loss?

11. Give an illustration by means of the number scale to show that a number may represent either a *change of position* in one direction or the other, or a *fixed position* with respect to the zero point.

12. Why do we call positive and negative numbers signed numbers? What is meant by the absolute value of a number?

13. State Principle IX in full.

14. By means of the number scale describe the "counting" method of adding signed numbers

15. Make a list of pairs of opposite qualities to which positive and negative numbers apply. State in each case what is represented by positive and what by negative numbers.

16. How is the correctness of subtraction tested in arithmetic? Is the same test applicable to subtraction in algebra?

17. Explain subtraction by counting on the number scale.

18. How do negative numbers make subtraction possible in cases where it is impossible in arithmetic?

19. What is a convenient rule for subtracting signed numbers? State Principle X.

20. Give examples of equations which could not be solved without negative numbers and show that such equations can be solved by means of negative numbers.

21. Give an example in which positive and negative numbers are multiplied. State Principle XI.

22. Define division. How do we obtain the law of signs in division? State Principle XII. What is the test of the correctness of division?

23. What may be the significance of a negative number when obtained as a result of solving a problem?

24. Explain how one set of signs + and − can be used to indicate both quality and operation.

Show that $a + {}^{-}b = a - b$ and $a - {}^{+}b = a - b$.

25. What is a polynomial? A term? How are polynomials classified? What are similar terms? By what principle are similar terms added? By what principle are they subtracted? In adding or subtracting polynomials, how may the terms be arranged for convenience? What is the principle for removing a parenthesis when preceded by the sign +? By the sign −? State Principle VII in full.

26. Make a diagram to show how to multiply $(7+4)$ by $(11+8)$ without first uniting the terms of the binomials. Multiply $(a+b)$ by $(c+d)$ in the same manner. Multiply $(12-3)$ by $(9-7)$ in two ways and compare results. State the principle by which two polynomials are multiplied.

27. Explain why x^2 is called the square of x or x squared. State in words what is the square of the binomial $(x+a)$; of the binomial $(x-a)$.

28. Show by an example how negative numbers may be used in solving a problem, even though the answer to the problem is positive and the statement of the problem does not involve negative numbers.

REVIEW EXERCISES

In the following exercises perform the indicated operations, remove all parentheses, solve all equations, verify the results, and in each case state the principles used.

1. Add $3x+4y-3z$, $5x-2y-z$, and $3y-5x+7z$.

2. From $15a+4b-13bc$ subtract $3a-8b+2bc$.

3. Subtract $7x-5y-7a$ from $6x+5y+3a$.

4. $(3x-4y-z)(x+y+z)$. 5. $(b-5)(2a^2b-3ab-b)$.

6. Add $11axy+13x-14y$, $2y-4x$, and $3y+x-8axy$.

7. $(5x-3b)+(2x+b)-(4x-2b-x+5b)$.

8. Add $19b+3c$, $2b-7c$, $2c-14b$, and $c+8b$.

9. $-(a-3b-c)-(2c-a-5b)+(a-c+b)$.

10. Subtract $2x+4y+z$ from $13x-3y-5z+8$.

11. $\frac{x}{3}+\frac{x}{6}+\frac{x}{12}+\frac{15}{4}=x$.

12. $5(x-7)-3(14-x)+60=1-10x$.

13. $13(1-x)-6(2x-5)=80+12x$.

14. $\frac{x-3}{7}+\frac{x-4}{5}+\frac{x-8}{2}+\frac{x-6}{6}=18.$

15. Add $7x-3y-4$, $5x+2y+5$, and $3y-8x-6$.

16. Add $13a+4b-9c$, $2c-8b-16a$, and $8a-5b-8c$.

17. $(13-4-2)(5x-3+4x)$.

18. $5(x-y)^2-5(y-x)^2+(x-y)(x+y)$.

19. Square 73, 79, 92, 98, 1003, and 995 as binomials.

20. From $17b-4a-2c-19$ subtract $8c-5a-8b+4$.

21. $3-(3-2+6+8-3)+8-(9-3+8)$.

22. $3(4-x)-2(5-6x)=8x+4$.

23. $\frac{x}{4}+\frac{x}{8}+\frac{x}{16}-\frac{x}{2}+\frac{x}{32}=-2.$

24. $\frac{y-3}{4}-\frac{y+9}{10}=\frac{y-11}{4}-2.$

25. $\frac{y-4}{3}+\frac{y+2}{3}+\frac{y+8}{3}=2y-20.$

26. $5x-(3x-2+2y+x)+13y-(6-3x+4)$.

27. From $3-4a-5c+8x^2$ subtract $2x^2-2a-4c+8$.

28. $12+(2a-3c-4b)-(3b-c-a-8)$.

29. $\frac{y}{2}+\frac{y+20}{4}+\frac{y+5}{5}=25.$

30. $\frac{y}{3}-\frac{y+20}{5}+\frac{y-5}{5}+\frac{y-10}{2}=15.$

31. Add $y-20$, $4y+6$, $2y+4x-13$, and $2x-8y-40$.

32. $(x-1)(2x-2)+(x-5)^2=(3-x)(24-3x)-7$.

33. Subtract $16-x+2z-4y$ from $3x-5z-8y$.

34. $19+(2x-7)-(31-4x-8-2x)=5x+7$.

35. $(4ab-6ac-5ad)(b-c+d)$.

36. $(17x+3)(x-1)+8=(2-x)(6-17x)+19$.

37. $5-(a+b-c-d+8)+(3+a+c-d)-5.$

38. $\frac{a}{10}+\frac{a+10}{10}+\frac{a-10}{10}=a-28.$

39. Add $6a+9$, $8a-13$, $46a-8$, and $6-54a$.

40. $(a-2)(6a-4)+2(a-1)^2=(6-a)(30-8a)+4.$

41. From $6(a+2)+3(c+4)-2(b-d)$ subtract $2(a+2)-2(c+4)+3(b-d)$.

42. $\frac{a}{3}+\frac{a+7}{4}-\frac{a-3}{3}=\frac{a+227}{5}-1.$

43. $(a-2-3c-8+2b)(6-a-c-b+8).$

44. $\frac{a-1}{2}+\frac{a+1}{2}+\frac{a-3}{12}=2+a.$

45. $a^2b-(3b-8a^2-7)+3ab^2-(4ab^2+8-2a^2).$

46. $\frac{a+1}{4}+\frac{a-3}{4}+\frac{a-7}{4}=2a-26.$

47. Add $12a^2b^2c+8ax$, $6ax-8a^2b^2c$, and $2ax+3a^2b^2c$.

48. Add $5xy^2+3x^2y+4xy$, $2x^2y-6xy^2-3xy$, and $4xy$.

49. Add $6ab-3c-2a$, $2c-4ab-5a$, $5c-a+ab$, and $3+5a-2c-3ab$.

50. $\frac{n+1}{3}+\frac{n+3}{4}+\frac{n-1}{4}=\frac{n+13}{3}+\frac{n-2}{3}.$

51. $(n-4)(6-3n)-(6-n)^2-10=-4n(n-4).$

52. From $35ab-8x-9z+13$ subtract $16ab-4z+5x+8$.

53. $(n+2)^2+(n-1)^2+(n+1)^2=3n(n+2)+60n+130.$

54. Subtract $5a-8x-6y$ from $13x+14y-15z-4a$.

55. From $9y-4x-6z-3b$ subtract $8-9y-3x-2z$.

56. $2x+4-6(5x-8-7x)+2-4x=6(2-3x)-42.$

57. $-(7+4x-8-2x)+4-2x=6x+25.$

58. $16 + 5x - (8x + 9 - 4x + 17) = 8x - 3$.

59. $6x - 3 - (4x + 8 - 9x) - (5x - 2) = x + 11$.

60. $(a - 1 + b - c - d)(4a + 5b + 3c - 2d)$.

61. $(4ax - 3ay + 5az - 8)(x + y - z + 2)$.

62. $(3ac - 2ad + 4ed)(2 + 3 + 4 + 5)$.

63. $7 - (3a - 2b - 4a) + b + 2a - (3b - 2a - a)$.

64. $8x + (5y - 5) + (2y - 1) - (13y + 8x - 17)$.

65. $16ax + 4 - (8 - 8ax - a) - (12ax - 13 - ax)$.

66. Add $15ax^2 + 3bc^2$, $2bc^2 - 7ax^2$, and $5 + 2ax^2 - 5bc^2$.

67. Add $16 - 7ab - 2a^2 + 5ab$, $4a^2 - 2ab$, and $5ab - 8$.

68. Add $51x^2y - 35 + 12a^2$, $41 - 17a^2 - 57x^2y$, and $3x^2y$.

69. Add $35b^2 - 13c^2$, $8c^2 - 3b^2c^2$, and $6b^2 - 8c^2 - 9b^2c^2$.

70. Add $19 - 2x + 3x^2b + 2b$, $4x^2b + x + 5b + 8$, and $4x$.

71. Subtract $2a - 6x^2b - 3x + 21$ from $19 - 2x + 3x^2b - 7a$.

72. From $6a - 45 + 8b - 3c + 82cb$ subtract $7b + 18 + 6c$.

73. $(17 + 2a - 3b - 4bc)(2 - a + b - c)$.

74. $(13c - 4d + 8e - 3)(c - d)$.

75. $(4xy - 2y - 3x - 2)(y - x + y + 5)$.

76. $(9ax - 3x - 5a - 2x + 4)(5 - x)$.

77. From $41a^2 + 7ab - 5c^2 - 9ab$ subtract $8ab + 7c^2 + 50a^2$.

78. From $15 + 2x - 9xy - 3y$ subtract $5xy - 4x - 2y$.

79. Subtract $12 + 8x - 4a - 6c - 18abc$ from $5x - 2a + 3c$.

80. $2 - (71x + 42y - 15x - 64) - (5 - 91x - 2y - 13xy)$.

81. $(31 - 2x + 3y - 5)(x + y)$.

82. $8x + (13 + 18x - 6) - (5 - 6x) = 16x + 10$.

83. $(9x-3)(4-x)+(x-3)^2=-8(x+2)^2+94.$

84. $(x+1)^2+(x+2)^2+(x+3)^2=(3x-1)(x+12)-43.$

85. $(2x+5)(x-7)-(x-1)^2=(x+1)(x+2)-28.$

86. $3(5-x)^2-(2x-1)(x-1)=(x-7)(x+10)+17x+50.$

87. $5-(7-4x+2y-4b)-(8x-6y-9+2b)+8a.$

88. $8x-(-3-2-4-7)+5x+(2+6+4)-(-3x+2).$

89. $5y+2x-(6-4x-5x)-3y-(4x-2y)-(-7y+8).$

90. $35y-(41x-16-12y)+5x+(-6+46y-18x).$

91. $(x-1)^2-(x-8)(2x-1)=-x^2+98.$

92. $(32+x)(4x-1)+(5-x)^2+(x-1)^2=6(x+1)^2+194.$

93. $(2x-7)(5-x)-(2-5x)(1-x)=-x(7x-34)-17.$

94. $(7+x)(x-4)+(1-x)^2=-23+2x^2.$

95. $(12-4x)(2-x)-4(1+x)^2=5x+119.$

96. $(x-17)(59-2x)-(1-x)^2=(6-3x)(x-2)+384.$

97. $(3x-2)+(x-1)^2+(x-2)^2=2(x-1)(x-2)+5.$

98. $(6-3x)(2+x)+16(x-1)^2=13(x+4)^2+364.$

99. $\dfrac{x+8}{2}-\dfrac{x-9}{12}+\dfrac{x-17}{6}=\dfrac{4x-7}{2}+\dfrac{2x+6}{3}+\dfrac{5-31x}{12}.$

100. $\dfrac{3x-1}{6}-\dfrac{3x+3}{3}+\dfrac{x-1}{2}=\dfrac{x+5}{6}+4x-\dfrac{20}{3}.$

CHAPTER IV

THE SOLUTION OF PROBLEMS

90. Some of the advantages of algebra over arithmetic in solving problems have been pointed out in the preceding chapters. For instance, brevity and simplicity of statement secured through the use of letters to represent numbers; the translation of the words and sentences of problems into number expressions and equations; and the clear and logical solution of the equation, step by step.

91. A still greater advantage is set forth in the present chapter; namely, the opportunity offered by the symbols and processes of algebra to *summarize a whole class of problems* under one solution, called the **formula**, which is thereafter used to solve all problems of the class.

PROBLEMS INVOLVING INTEREST

92. A class of problems already within the pupil's experience will illustrate this point. The different cases of percentage or interest have been studied in arithmetic, and a large number of isolated problems have been solved according to the rules. In this instance, therefore, we proceed at once to summarize all of this work in a few short statements.

Let $p =$ any principal, *i.e.* a number of dollars at interest.
$i =$ the interest, *i.e.* the number of dollars accrued.
$r =$ the rate, to be expressed in hundredths.
$t =$ the time, to be expressed in years and fractions of a year.

Then the rule of arithmetic for finding the interest when the principal, rate, and time are given is

$$\textit{interest} = \textit{principal} \times \textit{rate} \times \textit{time},$$

i.e.
$$i = prt. \tag{1}$$

If in this equation $i=150$, $r=.05$, $t=6$, find p. If $i=190.5$, $p=635$, $r=.03$, find t. If $i=665$, $p=1000$, $t=17$, find r. Substitute other values for any three of these letters and find the value of the remaining letter.

Solve equation (1) for t in terms of i, p, and r; also for p in terms of i, r, and t, and for r in terms of p, t, and i.

It follows that if any three of the four numbers, p, r, i, and t, are given, the remaining one may be found.

Let the student state four rules of interest by translating these formulas into words.

Note the simplicity of these equations compared with the corresponding rules in arithmetic.

Solve each of the following problems by substituting in the proper formula:

1. What is the simple interest at 5 % on \$400 for 5 years and 9 months?

Solution. $i = p \cdot r \cdot t = 400 \cdot \frac{5}{100} \cdot 5\frac{3}{4} = 4 \cdot 5 \cdot \frac{23}{4} = 115.$

2. In what time will the simple interest on \$750 at 3 % amount to \$225? Substitute in the formula $t = \dfrac{i}{pr}$.

3. What is the semi-yearly income from an endowment of \$2,700,000, the rate being $4\frac{3}{4}$ % per annum? (Here $t = \frac{1}{2}$.)

4. A father invested \$1500 at $5\frac{1}{3}$ %, the simple interest on which was to go to his eldest son on his 21st birthday. The young man received \$1240. How old was the son when the investment was made?

5. What is the amount of money invested if it yields \$787.50 interest per annum, the rate being $5\frac{1}{4}\%$? (Here $t=1$.)

6. A certain investment yields \$8160 in 8 years. What is the principal, if the rate of interest is 4%?

7. A \$45,800 investment yielded \$13,396.50 interest (simple) in $6\frac{1}{2}$ years. What was the rate of interest?

8. The endowment of a small college is \$750,000, the yearly income from which is \$45,000. What is the average rate at which the endowment is invested?

9. A capitalist has investments amounting to \$360,000, the total income from which amounts to \$1800 per month. What is the average rate at which the money is invested? ($t=\frac{1}{12}$.)

10. If \$700 is invested at $5\frac{1}{2}\%$ simple interest, what is the amount at the end of 5 years 6 months? This problem calls for the amount, which is the sum of principal and interest.

If $a=$ amount, then $\boldsymbol{a=p+i=p+prt}$.

11. Solve the equation $a=p+prt$ for p in terms of a, r, and t, and translate the result into words.

12. Solve $a=p+prt$ for r in terms of a, p, and t, and translate the result into words.

13. Solve $a=p+prt$ for t in terms of a, p, and r, and translate the result into words.

14. Seven years ago I invested a certain sum of money at 6% simple interest. The amount at present is \$5680. How much did I invest? (Use the formula obtained in Problem 11.)

15. Six years ago A invested \$470 at simple interest. The amount at present is \$611. What is the rate per cent?

16. Some years ago I invested \$500 at 7% simple interest, which now amounts to \$815. How many years ago was the investment made?

17. A merchant bought goods for \$250, and some months later sold the goods for \$300, making a profit of 1% per month. How many months between the purchase and the sale?

18. A real estate dealer sold a house and lot for \$7500, for which he received a commission of 4%. What was the profit?

In this case the element of time does not enter. The word *commission* is then used for interest and the principal is called the *base*. Letting $c =$ commission, $b =$ base, and $r =$ rate, we have

$$\boldsymbol{c = br.}$$

Applying this formula, $c = br = 7500 \cdot \frac{4}{100} = 75 \cdot 4 = 300.$

19. Solve the equation $c = br$ for b in terms of c and r, and translate the result into words.

20. Solve $c = br$ for r in terms of c and b, and translate the result into words.

21. A merchant's commission for selling a carload of peaches was \$18.75. What was the rate of commission if the peaches brought \$375? (Use the formula found in 19.)

22. A broker received \$25.50 commission for selling \$850 worth of bonds. What was his rate of commission?

23. An agent received 3% commission for buying and $3\frac{1}{2}$% for selling some property. He paid \$5750 for it and sold it for \$7200. What was his total commission?

24. How much must I remit to my broker in order that he may buy \$600 worth of bonds for me and reserve 5% for his commission?

I must send him both base and commission. Calling this the amount and representing it by a, we have

$$\boldsymbol{a = b + c = b + br = b(1 + r).}$$

Hence $\qquad a = 600 + 600 \cdot \frac{5}{100} = 600 + 6 \cdot 5 = 630.$

25. Solve the equation $a = b + br$ for b and translate the result into words.

26. Solve $a = b + br$ for r and translate the result into words.

27. A merchant received \$918 with which to buy corn after deducting his commission of 2% on the price of corn. How much was his commission and how much was used to buy corn?

Here $a = 918$. Find b by the formula under 25, which gives the sum paid for corn.

28. A broker received \$790, of which he invested \$750 in stocks, reserving the balance as his commission. Find the rate of his commission by means of the formula obtained in 26.

29. An agent received \$945 with which to buy lumber after deducting his commission of 5% on the cost of the lumber. How much was his commission? (First find the base by substituting in the formula of 25.)

30. A dealer sold berries for \$18.95, and after deducting a commission of 2% sent the balance to the truck gardener. How much did he remit?

The sum he sent was the difference between the base and the commission; calling this d, we have

$$d = b - c = b - br = b(1 - r).$$

Hence in this case $d = 18.95\left(1 - \frac{2}{100}\right) = 18.57$.

31. Solve the equation $d = b(1 - r)$ for b in terms of d and r and translate the result into words.

32. Solve the equation $d = b - br$ for r in terms of b and d and translate the result into words.

33. After deducting a commission of 3% for selling bonds, a broker forwarded \$824.50. What was the selling price of the bonds? (Solve by means of the formula under Problem 31.)

34. A broker sold stocks for \$1728 and remitted \$1693.44 to his principal. What was the rate of his commission? (Solve by means of the formula under Problem 32.)

35. In how many years will \$200 double itself at 5%?

In this case $i = p$. Hence, using formula (1), page 101, we have

$$200 = 200 \times \tfrac{5}{100} \times t.$$

Hence $$t = 20.$$

36. In how many years will any sum, p, double itself at any rate, r?

Here $p = prt$. Hence, solving, $t = 1 \div r$.

37. In what time will a sum of money double itself at 6%? 7%? $4\frac{1}{2}$%? $3\frac{2}{5}$%?

38. Collect all the formulas worked out in this set of problems. Translate each into words. Which were used as the original ones from which the others were deduced? How many and which ones are needed in order to derive all the others? Why are such formulas better than rules expressed in words?

PROBLEMS INVOLVING AREAS

93. Another class of problems already well known to the pupil in arithmetic concerns the areas of rectangles and triangles.

If in a rectangle we let the number of units of length be denoted by l, the number of units of width by w, and the number of square units in the area by a, then *area = length × width;*

i.e. $$a = lw. \tag{1}$$

If in equation (1) $a = 144$, $l = 16$, find w. If $a = 1116$, $w = 31$, find l. Substitute other values for any two of these letters and find the value of the remaining one.

Solve this equation for w in terms of l and a, and also for l in terms of w and a.

Also if b is the base of a triangle, h its altitude (height), and a its area, then *area* $= \frac{1}{2}$ (*base* $\times$ *altitude*);

i.e.
$$a = \frac{bh}{2}. \tag{2}$$

Substitute particular values for any two of these letters and find the value of the remaining one.

Solve (2) for h in terms of a and b, and also for b in terms of a and h.

Let the student translate each of these equations into words. Use these formulas in the solution of the following problems:

1. How many tiles each 3 by 4 inches are needed to cover a rectangular floor 18 by 22 feet? Use formula (1).

2. How long a space 25 feet in width can be covered by 340 square feet of roofing?

3. The base and altitude of a triangle are 8 and 6 respectively. Find its area.

4. The base of a triangle is 12 and its area 72. Find its altitude.

5. The altitude of a triangle is 16 and its area 144. Find its base.

6. A rectangle is 5 feet longer than it is wide. If it were 3 feet wider and 2 feet shorter, it would contain 15 square feet more. Find the dimensions of the rectangle.

Let w equal the width; then construct number expressions for the length, width, and area under the supposed conditions.

7. A rectangle is 6 feet longer and 4 feet narrower than a square of equal area. Find the side of the square and the sides of the rectangle.

8. The base of a triangle is 8 inches greater than its altitude. If the base is increased by 4 inches and the altitude decreased by 2 inches, the area remains unchanged. Find the base and altitude of the triangle.

9. A rectangle is 14 inches longer than it is wide. If the width is increased by 5 inches and the length decreased by 4 inches, the area is increased by 70 square inches. Find the dimensions of the rectangle.

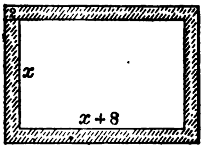
10. A rectangle is 15 rods longer and 10 rods narrower than an equivalent square. What are the dimensions of the rectangle?

11. The altitude of a triangle is 16 inches less than the base. If the altitude is increased by 3 inches and the base by 2 inches, the area is increased by 52 square inches. Find the base and altitude of the triangle.

12. The width of a rectangular field is 20 rods less than its length. If each side is decreased by 10 rods, the area is decreased by 900 square rods. What are the dimensions of the field?

13. A picture is 4 inches longer than it is wide. Another picture, which is 12 inches longer and 6 inches narrower, contains the same number of square inches. Find the dimensions of the pictures.

14. A picture, not including the frame, is 8 inches longer than it is wide. The area of the frame, which is 2 inches wide, is 176 square inches. Find the dimensions of the picture.



15. A picture, including the frame, is 10 inches longer than it is wide. The area of the frame, which is 3 inches wide, is 192 square inches. What are the dimensions of the picture?

16. The base of a triangle is 11 inches greater than its altitude. If the altitude and base are both decreased 7 inches, the area is decreased 119 square inches. Find the base and altitude of the triangle.

17. The base of a triangle is 3 inches less than its altitude. If the altitude and base are both increased by 5 inches, the area is increased by 155 square inches. Find the base and altitude of the triangle.

18. A commander in attempting to draw up his men in the form of a solid square finds that there are 80 men more than enough to complete the square. If he places 2 more men on each side of the square he needs 84 more men to complete it. How many men in his command?

PROBLEMS INVOLVING VOLUMES

94. Still another class of problems for which the fundamental formulas are already well known concerns the volumes of rectangular solids and of pyramids.

If the number of units of length, width, and height of a rectangular solid be denoted by l, w, and h respectively, and the number of units of volume by v, then

$$\textit{volume} = \textit{length} \times \textit{width} \times \textit{height};$$

i.e.
$$v = lwh. \tag{1}$$

If w is the width, l the length of the rectangular base of a pyramid, h its altitude, and v its volume, then

$$\textit{volume} = \tfrac{1}{3}\,(\textit{area of base} \times \textit{altitude});$$

i.e.
$$v = \frac{lwh}{3}. \tag{2}$$

In equations (1) and (2) substitute particular values for any three of the letters and find the value of the remaining one.

Use these formulas in solving the following problems:

1. Solve equations (1) and (2) for l, w, and h respectively and translate each equation into words.

2. How many cubic feet of earth are removed in digging a cellar 18 feet long, 12 feet wide, and 9 feet deep? Solve by substituting in formula (1).

3. If a cut in an embankment is 500 yards long and 4 yards deep, how wide is it if 18,760 cubic yards are removed?

4. How deep is a rectangular cistern which holds 500 cubic feet of water if it is 6 feet wide and 8 feet long?

5. The base of a pyramid is 16 inches long and 12 inches wide. Its altitude is 30 inches. Find its volume.

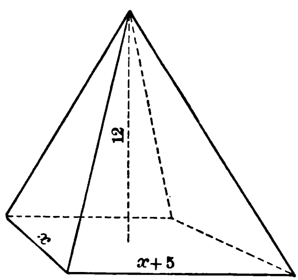
6. A pyramid whose volume is 72 cubic feet has a base whose area is 24 square feet. Find the altitude.

7. A pyramid whose volume is 91 cubic inches has an altitude of 21 inches. Find the area of its base.

8. A rectangular room which is 10 feet high is 4 feet longer than it is wide. If it were 5 feet longer and 2 feet wider, it would contain 950 cubic feet more than it does. Find its length and width.

9. A city building 50 feet high extends back 25 feet more than its frontage. If the building were 8 feet wider and extended 10 feet farther back, its capacity would be increased 59,000 cubic feet. What are the ground dimensions of the building?

10. A pyramid whose altitude is 12 inches has a rectangular base 5 inches longer than it is wide. If the length of the base is decreased by 1 inch and the width increased by 2 inches, the volume of the pyramid is increased by 72 cubic inches. Find the dimensions of the base of the pyramid.



11. The base of a rectangular pyramid whose altitude is 15 inches is 12 inches longer than it is wide. If the length and width of the base are both decreased by 3 inches, the volume is decreased by 675 cubic inches. Find the dimensions of the base of the pyramid.

12. The base of a rectangular pyramid is 15 inches wide. The altitude of the pyramid is 4 inches less than the length of the base. If the altitude is increased by 6 inches and the length of the base by 3 inches the volume is increased by 1155 cubic inches. Find the altitude of the pyramid and the length of its base.

13. The altitude of a pyramid is 7 inches less than the length of the base. The width of the base is 13 inches. If the altitude and the length of the base are both decreased by 3 inches, the volume is decreased by 364 cubic inches. Find the altitude of the pyramid and the length of its base.

14. The width of the base of a pyramid is 2 inches greater than its altitude. The length of the base is 34 inches. If the altitude and the width of the base are both increased by 6 inches, the volume is increased by 2992 cubic inches. Find the altitude and the width of the base.

PROBLEMS INVOLVING SIMPLE NUMBER RELATIONS

95. In the problems thus far considered in this chapter the formulas are already known from previous study or experience. Algebra affords a means of deriving such formulas in many cases where they are not already known.

1. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Let $g =$ the greater number, then $35 - g =$ the lesser.

2. The sum of two numbers is 48 and their difference is 24. What are the numbers?

3. The sum of two numbers is $41\frac{1}{2}$ and their difference is $23\frac{1}{2}$. What are the numbers?

4. The sum of two numbers is 8590 and their difference is 3480. What are the numbers?

The four problems just preceding are similar in character. A simple rule can be found for solving all problems of this kind. Consider problem 1.

We have	$g-(35-g)=5.$	(1)
By VII,	$g-35+g=5.$	(2)
By I, *A*,	$2g=35+5.$	(3)
By *D*, VI,	$g=\frac{35}{2}+\frac{5}{2}.$	(4)
By *F*,	$g=20$, the greater number,	(5)
and	$l=35-g=15$, the lesser number.	(6)

The results in the final form tell nothing more than the answers to this particular problem; but the value of g in the form $g=\frac{35}{2}+\frac{5}{2}$, if examined closely, tells much more. Stated in full it means:

$$\text{the greater} = \frac{\text{the sum of the numbers}}{2} + \frac{\text{their difference}}{2}$$

Examine now the solution of the other three problems to see whether this same expression will give the greater number in each case.

The great advantage in not adding 35 and 5 before dividing by 2, is that the expression $\frac{35}{2}+\frac{5}{2}$ preserves the original numbers as given in the problem, so that we see how each enters into the result.

If the sum of the two numbers is called s and their difference d, then we are *compelled* to keep these letters separate to the end of the solution.

Thus, the lesser is	$l=s-g,$
and	$g-(s-g)=d.$
By VII,	$g-s+g=d.$
By I, *A*,	$2g=s+d.$

Hence by D, VI, the greater is $g = \frac{s}{2} + \frac{d}{2}$, and the lesser is

$$l = s - g = s - \left(\frac{s}{2} + \frac{d}{2}\right) = s - \frac{s}{2} - \frac{d}{2} = \frac{2s}{2} - \frac{s}{2} - \frac{d}{2} = \frac{s}{2} - \frac{d}{2}.$$

Hence $$g = \frac{s}{2} + \frac{d}{2}, \text{ and } l = \frac{s}{2} - \frac{d}{2}.$$

These results put into words are as follows:

The greater of any two numbers is half their sum plus half their difference, and the lesser is half their sum minus half their difference.

Any problem of this kind is solved by substituting in this formula the particular values given to s and d and simplifying the results.

E.g. In problem 4, page 110, $s = 8590$ and $d = 3480$. Hence the greater number is $\frac{8590}{2} + \frac{3480}{2} = 4295 + 1740 = 6035$, and the lesser number is $\frac{8590}{2} - \frac{3480}{2} = 4295 - 1740 = 2555$.

5. Find by this formula two numbers whose sum is 17,540 and whose difference is 11,240.

6. Find two numbers whose sum is 40 and whose difference is 52. Solve also without the formula.

Evidently one of these must be a negative number in order to make the difference more than the sum, but the formula applies even in such cases.

7. Find two numbers whose sum is 38 and whose difference is 50. Solve also without the formula.

8. The sum of two numbers is 48, and one is 3 times the other. Find the numbers.

9. The sum of two numbers is 168, and one is 6 times the other. Find the numbers.

10. The sum of two numbers is s, and one is k times the other. Find the numbers.

Let $\quad n =$ one of the numbers.

Then $\quad s - n =$ the other number,

and $\quad kn = s - n.$

By A, $\quad kn + n = s.$

By I, $\quad (k+1)n = s.$

By D, $\quad n = \dfrac{s}{k+1}$, one of the numbers,

and $\quad kn = k \cdot \dfrac{s}{k+1}$, the other number.

Problems 8 and 9 may be solved by substitution in this formula. State this formula in words.

11. The sum of two numbers is 195, and one is 14 times the other. Find the numbers. Solve also without the formula.

12. The sum of two numbers is 75, and one is $\frac{2}{3}$ of the other. Find the numbers. Solve also without the formula.

13. The sum of two numbers is -52, and one is 12 times the other. Find the numbers. Solve also without the formula.

14. If 9 be added to a number and the sum multiplied by 4, the product equals 7 times the number. What is the number?

15. If 21 be added to a number and the sum multiplied by 5, the product equals 12 times the number. What is the number?

16. If a be added to a number and the sum be multiplied by b, the product is c times the number. What is the number?

If $n =$ the number. Show that

$$n = \frac{ab}{c-b}.$$

17. If 24 be added to a number and the sum multiplied by 3, the product is 9 times the number. Find the number. Solve by use of the formula of 16, and also without it.

18. If 3 be added to a number and the sum multiplied by 16, the product is 10 times the number. Find the number by use of the formula, and also without it.

19. The sum of three numbers is 108. The second is 16 greater than the first, and the third 25 greater than the second. What are the numbers?

20. The sum of three numbers is 98. The second is 7 greater than the first, and the third is 9 greater than the second. What are the numbers?

21. The sum of three numbers is s. The second is a greater than the first, and the third is b greater than the second. What are the numbers?

If $n =$ the first number, show that $n = \dfrac{s - 2a - b}{3}$.

Translate the result of problem 21 into words. It should be realized that if in problems 19 and 20 the numbers had been kept uncombined, as they were necessarily in 21, those results would translate into words, exactly as in 21.

Use the formula derived in 21 to solve the following, and also solve each one without the formula.

22. The sum of three numbers is 198. The second is 28 larger than the first, and the third is 25 larger than the second. What are the numbers?

23. The sum of three numbers is 31. The second is 3 more than the first, and the third 2 less than the second. What are the numbers?

Since the third number is 2 less than the second, this means that b of the formula is negative; *i.e.* $b = -2$.

24. The sum of three numbers is 91. The second is 11 less than the first, and the third is 12 more than the second. What are the numbers?

25. The sum of three numbers is 69. The second is 9 less than the first, and the third is 6 less than the second. What are the numbers? (How does the formula apply in this case?)

26. Divide the number 248 into two parts, such that 7 times the first is 42 more than twice the second.

27. Divide the number 645 into two parts, such that 13 times the first part is 20 more than 6 times the other.

28. Divide the number a into two parts, such that b times the first part is c more than d times the second part.

If x is the first part, show that $x = \dfrac{ad + c}{b + d}$.

29. Translate the formula in 28 into words.

30. Divide the number 1240 into two parts such that 5 times the first is 200 more than the second. Solve by use of the formula, and also without it.

PROBLEMS INVOLVING MOTION

96. Problems like those in the preceding class are useful chiefly in cultivating skill in deducing formulas, and so making rules. Those, however, in this and most of the following classes are extremely important in themselves, because of the simple laws of nature which they exemplify, and of which they afford a wide range of application.

97. In scientific language the distance passed over by a moving body is called the **space**, and the number of units of space traversed is represented by s. The **rate** of motion, that is the number of units of space traversed in the unit of time, is called the **velocity**, and is represented by v. The number of units of time occupied is represented by t.

E.g. At a certain temperature sound travels 1080 feet per second. Hence, in 5 seconds it will travel $5 \cdot 1080 = 5400$ feet. In this case, $s = 5400$ feet, $v = 1080$ feet per second, $t = 5$ seconds.

We then have the formula

$$s = vt. \qquad (1)$$

Solve the equation $s = vt$ for t in terms of s and v, and for v in terms of s and t.

Translate each of these formulas into words.

In equation (1) give particular values to any two of the letters and find the value of the remaining one.

It is to be understood in all problems here considered that the velocity remains the same throughout the period of motion; *e.g.* sound travels just as far in any one second as in any other second of its passage.

1. If sound travels 1080 feet per second, how far does it travel in 6 seconds?

2. If a glacier moves 450 feet per year, how far does it move in 7 years?

3. If a transcontinental train averages 35 miles per hour, how far does it travel in $2\frac{1}{2}$ days? (Given $v = 35$, $t = 2\frac{1}{2} \cdot 24$, to find s.)

4. A hound runs 23 yards per second and a hare 21 yards per second. If the hound starts 79 yards behind the hare, how long will it require to overtake the hare?

If t is the number of seconds required, then by formula (1) during this time the hound runs $23\,t$ yards and the hare runs $21\,t$ yards. Since the hound must run 79 yards farther than the hare, we have: $23\,t = 21\,t + 79$.

5. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the liner overtake the freight?

6. A motor boat starts $7\frac{3}{8}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How long will it require the motor boat to overtake the sailboat?

7. A freight train running 25 miles an hour is 200 miles ahead of an express train running 45 miles an hour. How long before the express will overtake the freight?

8. A bicyclist averaging 12 miles an hour is 52 miles ahead of an automobile running 20 miles an hour. How soon will the automobile overtake him?

9. A and B run a mile race. A runs 18 feet per second and B $17\frac{1}{2}$ feet per second. B has a start of 30 yards. In how many seconds will A overtake B? Which will win the race?

10. Two objects, A and B, move in the same direction, A at v_1* feet per second and B at v_2 feet per second. If A has n feet the start, in how many seconds will B overtake him?

If t is the number of seconds, then during this time A moves v_1t feet and B moves v_2t feet. Since B must move n feet farther than A, we have

$$v_2t = v_1t + n. \tag{2}$$

The solution of (2) for t gives the time sought.

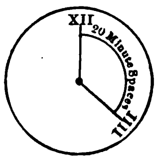
It is of the utmost importance that formulas (1) and (2) be clearly understood since they are fundamental in every motion problem in this chapter.

11. A fleet, making 11 knots per hour, is 1240 knots from port when a cruiser, making 19 knots per hour, starts out to overtake it. How long will it require?

12. In how many minutes does the minute hand of a clock gain 15 minute spaces on the hour hand?

Using one minute space for the unit of distance and 1 minute as the unit of time, the rates are 1 and $\frac{1}{12}$ respectively, since the hour hand goes $\frac{1}{12}$ of a minute space in 1 minute. Letting t be the number of minutes required, we have, just as in problem 10,

$$1 \cdot t = \tfrac{1}{12}\,t + 15.$$



13. In how many minutes after 4 o'clock will the hour and minute hands be together? (Here the minute hand must gain 20 minute spaces.)

* v_1 and v_2, read *v one* and *v two*, are used instead of two different letters to represent the velocities of the first and second respectively.

14. At what time between 5 and 6 o'clock is the minute hand 15 minute spaces behind the hour hand? At what time is it 15 minute spaces ahead?

Since, at 5 o'clock, it is 25 minute spaces behind the hour hand, in the first case it must gain $25 - 15 = 10$ minute spaces, and in the second case it must gain $25 + 15 = 40$ minute spaces. Make a diagram as in the preceding problem to show both cases.

15. At what time between 9 and 10 o'clock is the minute hand of a clock 30 minute spaces behind the hour hand? At what time are they together?

In each case, starting at 9 o'clock, how much has the minute hand to gain?

16. A fast freight leaves Chicago for New York at 8.30 A.M. averaging 32 miles per hour. At 2.30 P.M. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

If the express requires t hours to overtake the freight, the latter had been on the way $t + 6$ hours. Then the distance covered by the express is $55\,t$, and the distance covered by the freight is $32\,(t + 6)$. As these must be equal, we have $55\,t = 32\,(t + 6)$.

17. In a century bicycle race one rider averages $19\frac{1}{2}$ miles per hour, while another, starting 40 minutes later, averages $22\frac{1}{4}$ miles per hour. In how long a time will the latter overtake the former?

18. A sparrow flies 135 feet per second and a hawk 149 feet per second. The hawk in pursuing the sparrow passes a certain point 7 seconds after the sparrow. In how many seconds from this time does the hawk overtake the sparrow?

19. A courier starts from a certain point traveling v_1 miles per hour, and a hours later a second courier starts, going at

the rate of v_2 miles per hour. In how long a time will the second overtake the first, supposing v_2 greater than v_1?

If the second courier requires t hours to overtake the first the latter had been on the way $t + a$ hours. Thus the distance covered by the second courier is v_2t and by the first $v_1(t + a)$. As these numbers are equal we have

$$v_2t = v_1(t + a) \qquad (3)$$

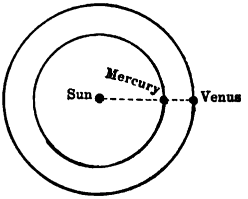
20. In an automobile race A drives his machine at an average rate of 53 miles per hour, while B, who starts $\frac{1}{4}$ hour later, averages 57 miles per hour. How long does it require B to overtake A? Use formula (3). Solve also by finding how far A has gone when B starts and then use formula (2).

21. A freight steamer leaves New York for Liverpool averaging $10\frac{1}{2}$ knots per hour, and is followed 4 days later by an ocean greyhound averaging $20\frac{1}{2}$ knots per hour. In how long a time will the latter overtake the former?

22. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together in the same direction, in how many seconds will the first gain one lap on the other? Two laps?

Let one lap be the unit of distance. Since the first covers one lap in 26 seconds his rate per second is $\frac{1}{26}$. Likewise the rate of the other is $\frac{1}{28}$. If t is the required number of seconds the distance covered by the first is $\frac{1}{26}t$ and by the second $\frac{1}{28}t$. If the first goes one lap farther than the second the equation is $\frac{1}{26}t = \frac{1}{28}t + 1$; if two laps farther it is $\frac{1}{26}t = \frac{1}{28}t + 2$.

23. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?



24. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, as in the figure, how long before they will again be in this position?

25. Saturn goes around the sun in 29 years and Jupiter in 12 years. Starting in conjunction, how soon will they be in conjunction again?

26. Uranus makes the circuit of its orbit in 84 years and Neptune in 164 years. If they start in conjunction, how long before they will be in conjunction again?

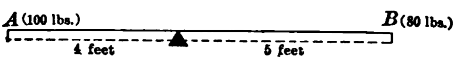
27. The hour hand of a watch makes one revolution in 12 hours, and the minute hand in one hour. How long is it from the time when the hands are together until they are again together?

28. One object makes a complete circuit in a units of time and another in b units (of the same kind). In how many units of time will one overtake the other, supposing b to be greater than a?

29. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

PROBLEMS INVOLVING THE LEVER

98. Two boys, A and B, play at teeter. They find that the teeter board will balance when equal products are obtained by multiplying the weight of each by his distance from the point of support.



Thus, if B weighs 80 pounds and is 5 feet from the point of support, then A, who weighs 100 lbs., must be 4 feet from this point, since $80 \times 5 = 100 \times 4$.

The teeter board is a certain kind of lever; the point of support is called the **fulcrum**.

In each of the following problems make a diagram similar to the above figure:

1. A and B weigh 90 and 105 lbs. respectively. If A is seated 7 feet from the fulcrum, how far is B from this point?

2. Using the same weights as in the preceding problem, if B is $6\frac{1}{2}$ feet from the fulcrum, how far is A from that point?

3. A and B are 5 and 7 feet respectively from the fulcrum. If B weighs 75 pounds, how much does A weigh?

4. A and B weigh 100 and 110 pounds respectively. A places a stone on the board with him so that they balance when B is 6 feet from the fulcrum and A $5\frac{1}{2}$ feet from this point. How heavy is the stone?

5. If the distances from the boys to the fulcrum are respectively d_1 and d_2, and their weights w_1 and w_2, then

$$d_1w_1 = d_2w_2. \tag{1}$$

If any three of these four numbers are given, the fourth may be found by means of this equation. Solve $d_1w_1 = d_2w_2$ for each of the four numbers involved in terms of the other three. Problems 1 to 4 can be solved by substitution in the formulas thus obtained.

6. A and B are seated at the opposite ends of a 13-foot teeter board. Using the weights of problem 1, where must the fulcrum be located so that they shall balance?

If the fulcrum is the distance d from A then it is $(13 - d)$ from B. Hence $90d = 105(13 - d)$.

7. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

8. A, who weighs 75 pounds, sits 7 feet from the fulcrum, and B, who weighs 105 pounds, sits on the other side. At what distance from the fulcrum should B sit in order to make a balance?

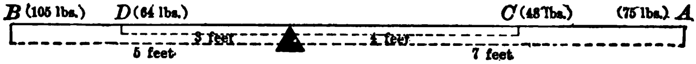
9. If in the preceding problem C, weighing 48 pounds, sits on the side with A and 4 feet from the fulcrum, where must D, who weighs 64 pounds, sit so as to maintain a balance?

From problem 8, when the teeter balanced it was found that A's weight acted like a lever with a downward force of $7 \cdot 75$ pounds and B's on the other side with a force of $5 \cdot 105$ pounds.

$$7 \cdot 75 = 5 \cdot 105.$$

Then in problem 9, C and D must sit so as to keep the board in balance, that is, so as to add the same downward force to both sides. Hence, as $4 \cdot 48$ is added on the side of A, $3 \cdot 64$ must be added on the side of B, since $4 \cdot 48 = 3 \cdot 64$. Therefore, adding these two equations, we still have the balance, namely,

$$7 \cdot 75 + 4 \cdot 48 = 5 \cdot 105 + 3 \cdot 64.$$



The weight of the boy multiplied by his distance from the fulcrum is called his **leverage**. The sum of the leverages on the two sides must be the same. Hence, if two boys, weighing respectively w_1 and w_2 pounds, are sitting at distances d_1 and d_2 on one side, and two boys, weighing w_3 and w_4 pounds, sitting at distances d_3, d_4 on the other side, then

$$\boldsymbol{w_1 d_1 + w_2 d_2 = w_3 d_3 + w_4 d_4}. \tag{2}$$

10. If two boys weighing 75 and 90 pounds sit at distances of 3 and 5 feet respectively on one side and one weighing 82 pounds sits at 3 feet on the other side, where should a boy weighing 100 pounds sit in order to make the board balance?

11. A beam carries a weight of 240 pounds $7\frac{1}{2}$ feet from the fulcrum and a weight of 265 pounds at the opposite end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

PROBLEMS INVOLVING DENSITIES

99. If a cubic foot of a certain kind of rock weighs 2.5 times as much as a cubic foot of fresh water, the **density** of this rock is said to be 2.5. If a cubic foot of oak weighs .85 times as much as a cubic foot of fresh water, the density of the oak is .85. The density of water is taken as the standard with which the densities of other substances are compared.

Thus, when we say that the density of a certain kind of iron is 7.25, we mean that any given volume of the iron weighs 7.25 times as much as a like volume of fresh water; and when we say that the density of cork is .24, we mean that a given volume of cork weighs .24 times as much as a like volume of fresh water.

A cubic centimeter of distilled water at the freezing point which weighs one gram is used as a standard of comparison. We therefore say that the density of any substance is equal to the number of grams which a cubic centimeter of it weighs.

E.g. if a cubic centimeter (ccm.) of a certain kind of marble weighs 2.5 grams, then it weighs 2.5 times as much as the same volume of water, and hence its density is 2.5.

The weight of an object in grams is, therefore, the product of its volume in cubic centimeters multiplied by its density.

E.g. if the density of cork is .24, this means that a cubic centimeter of cork weighs .24 grams. Any volume of cork, say 10 ccm., weighs $10 \times .24 = 2.4$ grams. Hence, if we represent the weight of an object in grams by w, its volume in cubic centimeters by v, and its density by d, we have the relation,

$$\boldsymbol{w = vd}. \tag{1}$$

1. What is the weight of an object whose density is 4.3 and whose volume is 250 ccm.? (Here $v = 250$, $d = 4.3$. Find w.)

2. What is the density of an object whose weight is 23.5 and whose volume is 17 ccm.?

3. What is the volume of an object whose weight is 24 grams and whose density is .65?

4. If 500 ccm. of alcohol, density .79, is mixed with 300 ccm. of distilled water, what is the density of the mixture?

The volume of the mixture is the sum of the volumes, and the weight of the mixture is the sum of the weights, of the water and alcohol. Hence, from formula (1):

$$500 \times .79 + 300 \times 1 = d \times 800. \quad \text{To find } d.$$

5. If 1200 ccm. of cork, density .24, are combined with 64 ccm. of steel, density 7.8, what is the average density of the combined mass? Will it float or sink? (A substance sinks if its density is greater than that of water.)

In solving problems of this kind find the expressions representing weight, density, and volume, and substitute in the equation $w = dv$.

6. How many cubic centimeters of cork, density .24, must be combined with 75 ccm. of steel, density 7.8, in order that the average density shall be equal to that of water, *i.e.* so that the combined mass will just float?

Let $v =$ volume of cork to be used. Then the total volume is $75 + v$, the total weight is $75 \times 7.8 + .24\,v$, and the density is 1. Hence, $75 \times 7.8 + .24\,v = 1 \cdot (75 + v)$. Solve this equation for v.

7. Brass is an alloy of copper and zinc. How many cubic centimeters of zinc, density 6.86, must be combined with 100 ccm. of copper, density 8.83, to form brass whose density is 8.31?

8. Coinage silver is an alloy of copper and silver. How many ccm. of copper, density 8.83, must be added to 10 ccm. of silver, density 10.57, to form coinage silver, whose density is 10.38?

9. The density of pure gold is 19.36 and of nickel 8.57. How many ccm. of nickel must be mixed with 10 ccm. of pure gold to form 14 karat gold whose density is 14.88.

10. How much mercury, density 13.6, must be added to 20 ccm. of gold, density 19.36, so that the density of the compound shall be 16.9?

11. What is the average density of 40 ccm. of water, density 1, and 180 ccm. of alcohol, density .79?

12. How many cubic centimeters of water must be mixed with 350 ccm. of alcohol, so that the density of the mixture shall be .97?

13. The density of copper is 8.83. 500 ccm. of copper is mixed with 700 ccm. of lead, whose density is 11.35. What is the density of the combined mass?

14. When 960 ccm. of iron, density 7.3, is fastened to 8400 ccm. of white pine, the combination just floats, *i.e.* has a density of 1. What is the density of white pine?

PROBLEMS ON MOMENTUM

100. The force with which a moving body strikes another depends both upon its mass and upon its rate of motion. The product of the mass and velocity of a moving body is called its **momentum**. The mass of a body is proportional to its weight. Hence weight is often used instead of mass.

What is the momentum of a body whose weight is 10 pounds and which moves 15 feet per second?

What is the velocity of a body whose weight is 50 and whose momentum is 350 pounds?

What is the weight of a body whose momentum is 500 and whose velocity is 25 feet per second?

A bullet weighing $\frac{1}{5}$ of a pound, moving 2250 feet per second, has a momentum equal to that of a stone weighing 50 pounds, which is hurled at the rate of 9 feet per second, since $\frac{1}{5} \cdot 2250 = 50 \cdot 9$.

By careful experiment it has been found that when a moving body strikes a body at rest but free to move, the two will move on with a combined momentum equal to the momentum of the first body before the impact.

Thus, if a freight car, weighing 25 tons and moving at the rate of 12 miles per hour, strikes a standing car weighing 15 tons, the two will move on with the original momentum of $12 \cdot 25$. But as the combined weight is now $25 + 15$, the *rate* of motion has been decreased to $7\frac{1}{2}$ miles per hour, since $12 \cdot 25 = 7\frac{1}{2}\,(25 + 15)$. In this case the automatic coupler connects the cars and they move on together. Even if the

two bodies after impact do not cling together, as two croquet balls, still the momentum of the one plus that of the other equals the original momentum.

E.g. if a croquet ball weighing 8 ounces and moving 20 feet per second, strikes another weighing 7 ounces and starts it off at the rate of 18 feet per second, then if the diminished velocity of the first ball is called v, we have

$$8 \cdot 20 = 7 \cdot 18 + 8v,$$

and solving, $v = 4.25.$

This indicates that the first ball is nearly stopped, which coincides with common observation.

1. In a switch yard a car weighing 40 tons and moving 8 miles per hour strikes a standing car weighing 24 tons. What is the velocity of the two after impact?

2. A billiard ball weighing 6 ounces and moving 16 feet per second strikes another ball which it sends off at the rate of 10 feet per second. The rate of the first ball is reduced to 9 feet per second by the impact. What is the weight of the second ball?

Since the momentum before impact equals the sum of the momentums after impact, we have $6 \cdot 16 = 9 \cdot 6 + 10w$, w being the unknown weight of the second ball.

3. A bowler uses a 16-ounce ball to take down the last pin. The ball sends the pin off at a velocity of 6 feet per second, the weight of the pin being 48 ounces, while the velocity of the ball is reduced to 4 feet per second. With what velocity did the ball strike the pin?

Since the momentum before impact equals the sum of the momentums after impact, we have $16v = 6 \cdot 48 + 4 \cdot 16$, v being the unknown velocity of the ball before impact.

4. In each of these problems we have considered the weight of two bodies, which we may call w_1 and w_2. If we call v_1 the velocity with which the first strikes the second, v_2 the velocity

imparted to the second, and v_1' the resulting decreased velocity of the first,* we have

$$w_1v_1 = w_1v_1' + w_2v_2 \qquad (1)$$

Translate this equation into words. It contains 5 different numbers, w_1, w_2, v_1, v_1', and v_2. If any four of these are given, the fifth may be found by solving this equation for that one in terms of the other four.

5. Solve the equation (1) for v_1 in terms of w_1, w_2, v_1', and v_2. Translate into words.

6. Solve the equation (1) above for w_1 in terms of v_1, v_1', w_2, and v_2. Translate into words.

7. Solve the equation (1) above for v_1' in terms of w_1, w_2, v_1, and v_2. Translate into words.

8. Solve equation (1) for w_2 in terms of w_1, v_1, v_1', v_2, and translate the result into words.

9. Solve equation (1) for v_2 in terms of w_1, w_2, v_1, v_1', and translate the result into words.

10. In the result of the last exercise substitute $w_1 = 50$, $w_2 = 40$, $v_1 = 10$, $v_1' = 2$, and find the value of v_2. Make a problem to fit this case.

11. In the result of problem 8 substitute $w_1 = 1000$, $v_1 = 75$, $v_1' = 25$, $v_2 = 50$, and find the value of w_2. Make a problem to fit this case.

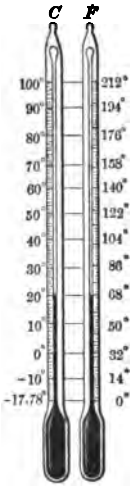
12. In the result of problem 6 substitute $v_1 = 60$, $v_1' = 10$, $w_2 = 80$, $v_2 = 25$, and find the value of w_1. Make a problem to fit this case.

13. In the result of problem 7 substitute $w_1 = 250$, $w_2 = 125$, $v_1 = 50$, and $v_2 = 50$, and find the value of v_1'. Make a problem to fit this case.

* v_1' is read *v one prime* and is used rather than a new letter for the purpose of recalling that this is *another* rate of the first body.

PROBLEMS ON THERMOMETER READINGS

101. There are two kinds of thermometers in use in this country, called the Fahrenheit and Centigrade, the former for common purposes, and the latter for scientific records and investigations. Hence it frequently becomes necessary to translate readings from one kind to the other.



The freezing and boiling points are two fixed temperatures by means of which the computations are made. On the Centigrade these are marked 0° and 100° respectively and on the Fahrenheit they are marked 32° and 212° respectively. See the cut. Hence between the two fixed points there are 100 degrees Centigrade and 180 degrees Fahrenheit.

That is, 100 degree spaces on the Centigrade correspond to 180 degree spaces on the Fahrenheit.

Hence 1° *Centigrade corresponds to* $\frac{9}{5}°$ *Fahrenheit,* or 1° *Fahrenheit corresponds to* $\frac{5}{9}°$ *Centigrade.*

All problems comparing the two thermometers are solved by reference to these fundamental relations.

1. If the temperature falls 15 degrees Centigrade, how many degrees Fahrenheit does it fall?

2. If the temperature rises 18 degrees Fahrenheit, how many degrees Centigrade does it rise?

3. Translate $+25°$ Centigrade into Fahrenheit reading.

25° Centigrade equals $\frac{9}{5} \cdot 25° = 45°$ Fahrenheit.

45° above the freezing point $= 45° + 32°$ above 0° Fahrenheit. Hence, calling the Fahrenheit reading F, we have $F = 32 + \frac{9}{5} \cdot 25$.

4. Translate $+14°$ Centigrade into Fahrenheit reading.

Reasoning as before, $F = 32 + \frac{9}{5} \cdot 14$.

From the two preceding problems we have the formula

$$F = 32 + \tfrac{9}{5}C. \qquad (1)$$

Translate this into words, understanding that F and C stand for readings on the respective thermometers.

5. Solve the above equation for C in terms of F and find

$$C = \tfrac{5}{9}(F - 32). \qquad (2)$$

Translate this into words. Verify the solution of each of the following by referring to the cut on page 128.

6. Translate $+41°$ Fahrenheit into Centigrade reading. Substitute in the proper formula.

7. Translate $+98°$ Fahrenheit, blood heat, into Centigrade reading.

8. Translate $-20°$ Fahrenheit into Centigrade reading.

9. Translate $-40°$ Centigrade, the freezing point of mercury, into Fahrenheit reading.

10. Translate $0°$ Fahrenheit into Centigrade by use of the formula. Also $0°$ Centigrade into Fahrenheit.

11. Translate $+212°$ Fahrenheit into Centigrade, and also $+100°$ Centigrade into Fahrenheit.

12. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is $102°$?

13. What is the temperature Fahrenheit when the sum of the Centigrade and Fahrenheit readings is zero?

14. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is $140°$?

15. What is the temperature in each reading when the Fahrenheit is $50°$ higher than the Centigrade?

16. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

PROBLEMS ON THE ARRANGEMENT AND VALUE OF DIGITS

102. If we speak of the number whose 3 digits, in order from left to right, are 5, 3, and 8, we mean $538 = 500 + 30 + 8$. Likewise, the number whose three digits are h, t, and u is written $100h + 10t + u$.

Hence, when letters stand for the digits of numbers written in the decimal notation, care must be taken to multiply each letter by 10, 100, 1000, etc., according to the position it occupies.

Illustrative Problem. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 18 greater than the first number. What is the number?

Solution. Let $x =$ the digit in tens' place.

Then $6 - x =$ the digit in units' place.

Hence, the number is $10x + 6 - x$. Reversing the order of the digits, we have as the new number $10(6 - x) + x$.

Hence $$10(6 - x) + x = 18 + 10x + 6 - x.$$

1. A number is composed of two digits, the digit in units' place being 2 greater than the digit in tens' place. If 4 is added to the number, it is then equal to 5 times the sum of the digits. What is the number?

2. A number is composed of two digits, the digit in tens' place being 3 greater than the digit in units' place. The number is one more than 8 times the sum of the digits. What is the number?

3. A number is composed of two digits whose sum is 9. If the order of the digits is reversed, we obtain a number which is equal to 12 times the remainder when the units' digit is taken from the tens' digit. What is the number?

4. A number is composed of two digits whose difference is 4. If the order of the digits is reversed, we obtain a number which is 3 less than 4 times the sum of the digits. What is the number?

5. The digit in tens' place is 1 more than twice the digit in units' place. If 36 is subtracted from the number, the order of the digits will be reversed. What is the number?

6. The digit in units' place is 2 less than twice the digit in tens' place. If the order of the digits is reversed, the number is unchanged. What is the number?

7. The digit in tens' place is 12 less than 5 times the digit in units' place. If the order of the digits is reversed, the number is equal to 4 times the sum of the digits. What is the number?

8. A number is composed of three digits. The digit in units' place is 3 greater than the digit in tens' place, which in turn is 2 greater than the digit in hundreds' place. The number is equal to 96 plus 4 times the sum of the digits. What is the number?

REVIEW QUESTIONS

1. In order that a problem may be solved by means of a formula, how many of the letters in the formula must be given by the problem? Illustrate this by the formula $i = prt$.

2. State the formulas involving areas and volumes which have been used in this chapter. Solve each formula for each of its letters in terms of all the others.

3. The area of a circle is found by squaring its radius and multiplying by 3.1416. State this rule as a formula, using the Greek letter π for 3.1416.

4. The volume of a circular column is found by multiplying the area of its base by its height. State this rule as a formula, using r for the radius of the base and h for the height.

5. State formulas for finding two numbers when their sum and difference are given.

Find a number such that if 16 be subtracted from it, 5 times this difference equals the number.

Construct other problems like this and then make a formula for the solution of all such problems. (See page 110.)

6. State three important formulas used in the solution of motion problems. Translate each into words. Solve each formula for each of its letters.

Of the motion problems, 4 to 29, which ones involve formula (1)? Which formula (2) and which formula (3)?

7. State the formulas used in this chapter in working problems on the lever. Solve each formula for each of its letters.

Using a teeter board 12 feet long with the fulcrum in the middle, how may three boys weighing 50, 75, and 100 lbs. respectively be seated on it so as to make the board balance? Is there more than one solution? Make a diagram for each of your results.

8. What is meant by the density of a substance? What is used as a standard (unit) of density with which the densities of other substances are compared?

What is the relation between the weight of a substance, its volume and its density?

9. Define momentum. When a moving car strikes a standing car, free to move, what can you say of the momentum of the two after impact?

10. Describe the Fahrenheit and Centigrade thermometers. State the formulas for the reading of each thermometer in terms of the other.

11. Write 347 as a trinomial. Write the number whose three digits in order are a, b, c; also the number whose three digits in order are c, b, a.

REVIEW PROBLEMS

103. Most of the following problems can be solved by substitution in some of the formulas developed in this chapter. Solve as many as possible in this way.

1. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

2. Divide the number 144 into two parts, so that $\frac{1}{4}$ of the greater is 9 more than $\frac{1}{5}$ of the smaller.

3. The sum of two numbers is 2890. Seven times one is 266 less than 5 times the other. What are the numbers?

4. What is the simple interest on \$400 at $6\frac{1}{2}\%$ for 7 years and 9 months?

5. The number of telegraph messages sent in the United States in 1905 was 5 million less than three times as great as in 1880, and 69 million less than twice that in 1900; while in 1900 it was 16 million more than twice as great as in 1880. Find the number of messages in each of these years.

6. The same number is added to each of the numbers 8, 9, 10, 12. What is this number if the product of the first and last sums is equal to the product of the second and third sums?

7. Find the time between 4 and 5 o'clock when the hands of the clock are 30 minute spaces apart.

8. A man buys a house for \$6500. His yearly tax on the property is \$57. The coal costs \$60 per year, repairs \$50, and janitor service \$108. To what monthly rental are his expenses equivalent if money is worth 5%?

9. A boatman rowing down a river makes 23 miles in 3 hours and returns at the rate of $3\frac{1}{2}$ miles per hour. How fast does the river flow?

10. A bird flying with the wind goes 65 miles per hour, and flying against a wind twice as strong it goes 20 miles per hour. What is the rate of the wind in each case?

11. A steamer going with the tide makes 19 miles per hour, and going against a current $\frac{1}{2}$ as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

12. A boatman trying to row up a river drifts back at the rate of $1\frac{1}{2}$ miles per hour, while he can row down the river at the rate of 12 miles per hour. What is the rate of the current?

13. A boatman rowing with the tide makes n miles per hour; rowing against a tide k times as strong, he makes m miles per hour. At what rate does he row, and what is the velocity of the stream? State the result as a formula.

14. A beam is 16 feet long. At what point must it be supported if it is to carry, when balanced, 460 pounds at one end and 690 at the other, its weight being disregarded?

15. Two numbers differ by 2 and the difference of their squares is 100. What are the numbers?

16. A beam is 12 feet long. It carries a 40-pound weight at one end, a 60-pound weight 3 feet from this end, and a 70-pound weight at the other end. Where is the fulcrum if the beam is balanced?

17. There is a number composed of 2 digits whose tens' digit is 2 less than its units' digit. The number is 1 less than 5 times the sum of its digits. What is the number?

18. A farm containing 240 acres can be rented at $3 per acre. The renter finds that if he borrows money at 5% to buy the farm he will save $125 per year. Find its value.

19. Two trains start from the same point at the same time and in the same direction, one making 25 miles per hour and the other 42 miles per hour. When will they be 238 miles apart?

20. The number of national banks in the United States on March 1, 1906, was 1322 less than twice as many as on March 1, 1900, and the number in 1903 was 1009 greater than in 1900. If the number in 1906 be subtracted from 3 times the number in 1903 the remainder is 7936. Find the number in each of the three years mentioned.

21. The earth and Mars were in conjunction July 12, 1907. When are they next in conjunction if the earth's period is 365 days and that of Mars 687 days? (See figure, page 119.)

22. $7500 is invested at 4% simple interest. Seven years and three months later the amount is used to build a house. What is the cost of the house if $735 has to be added to complete it?

23. There is a number composed of two digits whose sum is 12. If the order of the digits is reversed, the number is increased by 18. Find the number.

24. A slow steamer sails from New York to Liverpool, making 9 knots per hour. A swift liner follows 62 hours later, making $20\frac{3}{4}$ knots per hour. In how many hours will the latter overtake the former?

25. A man invested a certain sum of money at 5% simple interest. The amount $3\frac{3}{4}$ years later was $950. What was the investment?

26. The capital stock of the Bank of France is 35.6 million dollars less than that of the Bank of England and 6.3 million greater than that of the Imperial Bank of Germany. The combined capital stock of the three banks is 134.9 million dollars. Find the capital stock of each.

27. The capital stock of the Bank of Italy is 27.7 million dollars less than twice that of the Imperial Bank of Russia, and the capital of the Bank of Austria-Hungary is 13.6 million greater than that of the Bank of Russia. Their combined capital is 99.1 million dollars. Find the capital of each bank.

28. In a bicycle race *A* starts 32 minutes ahead of *B*. *B* rides at the rate of $20\frac{1}{3}$ miles per hour, while *A* rides $18\frac{2}{5}$ miles per hour. How many miles from the starting point does *B* overtake *A*?

29. A squadron of warships sails 13 knots per hour. A torpedo boat making $27\frac{2}{3}$ knots leaves port 19 hours later to overtake the squadron. In how many hours after leaving port will the torpedo boat overtake it?

30. A man takes out a life insurance policy for which he pays in a single payment. Thirteen years later he dies and the company pays \$12,600 to his estate. It was found that his investment yielded 2% simple interest. How much did he pay for the policy?

31. A merchant bought goods for \$600 and some months later sold them for \$648, making a profit of 2% per month. How many months elapsed between the purchase and the sale?

32. In a building there are at work 18 carpenters, 7 plumbers, 13 plasterers, and 6 hod carriers. Each plasterer gets \$1.90 per day more than the hod carriers, the carpenters get 35 cents per day more than the plasterers, and the plumbers 50 cents per day more than the carpenters. If one day's wages of all the men amount to \$183.45, how much does each get per day?

33. A train running 46 miles per hour leaves Chicago for New York at 7 A.M. Another train running 56 miles per hour leaves at 9.30 A.M. Find when the trains will be 15 miles apart. (Two answers.)

34. Divide the number 280 into two parts so that $\frac{2}{3}$ of one part is 48 less than $\frac{4}{5}$ of the other.

35. A merchant bought goods and sold them 5 months later for \$2687.50, making a gain on his investment of $1\frac{1}{2}$% per month. How much did he pay for the goods?

36. A bicyclist starts out riding 12 miles per hour, and is followed 40 minutes later by another riding 16 miles per hour. Find when they will be 5 miles apart. (Two answers.)

37. A father invested \$1000 at $6\frac{1}{2}$ % interest. When the principal and simple interest amounted to \$2235, it was given to the son for the expenses of his college education. How long had the money been invested?

38. Two trains start west at the same time, one from New York and the other from Philadelphia. If the New York train runs 55 miles per hour and the Philadelphia train 47 miles per hour, how long before they are 15 miles apart, the distance from New York to Philadelphia being 90 miles?

39. A man bought a tract of coal land and sold it a month later for \$93,840. If his gain was at the rate of 24 % per annum, what did he pay for the land?

40. There is a rectangle whose length is 60 feet more, and whose width is 20 feet less, than the side of a square of equal area. Find the dimensions of the square and the rectangle.

41. A number is composed of two digits whose sum is 14. If the digits are interchanged, the number is decreased by 18. What is the number?

42. What time between 7 and 8 o'clock are the hour and the minute hands of a clock together?

43. Change 104 degrees Fahrenheit (fever heat) to Centigrade reading.

44. A steamer leaves Liverpool for New York Saturday, 9 A.M., averaging 18 knots per hour. Seven hours later another steamer leaves New York for Liverpool, making $20\frac{1}{2}$ knots per hour. What time (Liverpool time) will the steamers meet if the trans-Atlantic distance by their course is 2940 knots?

CHAPTER V

INTRODUCTION TO SIMULTANEOUS EQUATIONS

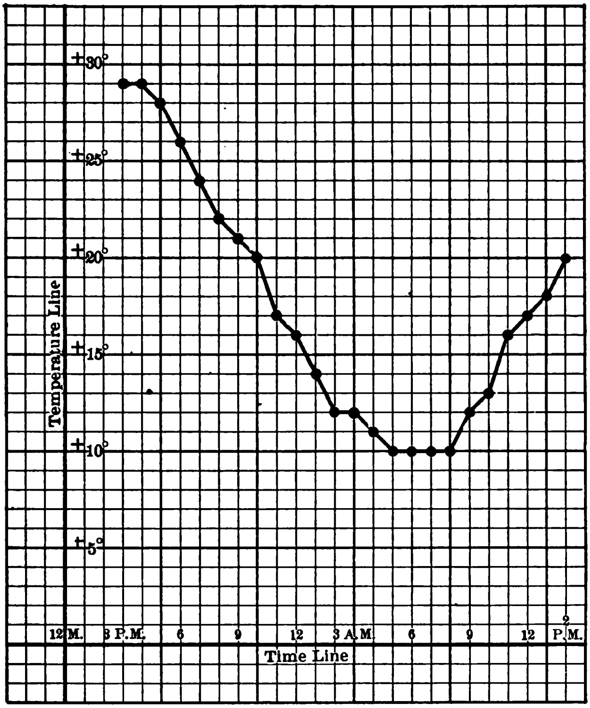
104. Graphic Representation of Statistics.

A graphic representation of the temperatures recorded by the U. S. Weather Bureau at Chicago on December 6 and 7, 1906, is shown on the next page. The readings were as follows:

3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

In the graph each heavy dot represents the temperature at a certain hour. The distance of a dot to the right of the heavy vertical line indicates the hour of the day counted from noon December 6th, and its distance above the heavy horizontal line indicates the thermometer reading at that hour. The lines joining these dots complete the picture representing the gradual changes of temperature from hour to hour.

Graphs of this kind are used in commercial houses to represent variations of sales, fluctuations of prices, etc. They are used by architects and engineers to show the comparative strength of materials, stresses to which they are subjected, and to exhibit multitudes of other data. They are used by the historian to represent changes in population, fluctuations in mineral productions, etc. In algebra they are used in solving many practical problems and in helping to understand many difficult processes. In the succeeding exercises the cross-ruled paper is essential.



EXERCISES

Make a graphic representation of each of the following tables of data:*

1. The population of the United States as given by the census reports from 1790 to 1900:

1790 . . 3.9 (million)	1830 . . 12.9	1870 . . 38.6
1800 . . 4.3	1840 . . 17.1	1880 . . 50.2
1810 . . 7.2	1850 . . 23.2	1890 . . 62.6
1820 . . 9.6	1860 . . 31.4	1900 . . 76.3

* In each case the number to be represented by one space on the cross-ruled paper should be chosen so as to make the graph go conveniently on a sheet. Thus in (1) let one small horizontal space represent two years and one vertical

2. The population of New York city since 1800:

1790 . . 33 (thousand)	1830 . . 202	1870 . . 942
1800 . . 60	1840 . . 312	1880 . . 1206
1810 . . 96	1850 . . 515	1890 . . 1530
1820 . . 123	1860 . . 813	1900 . . 1850

3. The population of Chicago since 1850:

1850 . . 30 (thousand)	1870 . . 306	1890 . . 1100
1860 . . 109	1880 . . 503	1900 . . 1698

4. The world's yearly production of gold since 1872:

1872 . . 99.6 (million)	1883 . . 102.4	1894 . . 181.5
1873 . . 96.2	1884 . . 101.7	1895 . . 194.0
1874 . . 90.7	1885 . . 108.4	1896 . . 202.3
1875 . . 97.5	1886 . . 106.0	1897 . . 236.1
1876 . . 103.7	1887 . . 105.8	1898 . . 286.9
1877 . . 114.0	1888 . . 110.2	1899 . . 306.7
1878 . . 119.0	1889 . . 123.5	1900 . . 254.6
1879 . . 109.0	1890 . . 118.9	1901 . . 262.5
1880 . . 106.5	1891 . . 130.7	1902 . . 296.0
1881 . . 103.0	1892 . . 146.3	1903 . . 325.5
1882 . . 102.0	1893 . . 157.2	1904 . . 346.8

5. The world's yearly production of silver since 1872:

1872 . . 65.0 (million)	1883 . . 115.3	1894 . . 214.5
1873 . . 81.8	1884 . . 105.5	1895 . . 208.0
1874 . . 71.5	1885 . . 118.5	1896 . . 203.0
1875 . . 80.5	1886 . . 120.6	1897 . . 207.0
1876 . . 87.6	1887 . . 124.3	1898 . . 218.6
1877 . . 81.0	1888 . . 140.7	1899 . . 217.6
1878 . . 95.0	1889 . . 155.4	1900 . . 224.0
1879 . . 96.0	1890 . . 163.0	1901 . . 223.7
1880 . . 96.7	1891 . . 177.0	1902 . . 208.6
1881 . . 102.0	1892 . . 197.7	1903 . . 220.4
1882 . . 111.8	1893 . . 209.1	1904 . . 217.7

space a million of population; and in (3) let one horizontal space represent one year and one large vertical space one hundred thousand of population.

6. Temperature, at Port Conger, New York, and Singapore. The average temperature for each half month is given.

Month		Port Conger	New York	Singapore	Month		Port Conger	New York	Singapore
Jan.	1–15	−35°	30°	80°	July	1–15	+32°	66°	86°
	16–31	−40°	28°	82°		16–31	+37°	68°	86°
Feb.	1–15	−45°	32°	84°	Aug.	1–15	+37°	68°	87°
	16–28	−40°	35°	84°		16–31	+34°	66°	86°
Mar.	1–15	−35°	38°	85°	Sept.	1–15	+27°	62°	85°
	16–31	−30°	40°	85°		16–30	+20°	60°	85°
April	1–15	−20°	45°	85°	Oct.	1–15	+ 8°	55°	85°
	16–30	−10°	48°	85°		16–31	− 2°	50°	84°
May	1–15	0°	50°	85°	Nov.	1–15	−15°	48°	83°
	16–31	+ 8°	56°	86°		16–30	−20°	45°	82°
June	1–15	+16°	60°	86°	Dec.	1–15	−28°	42°	82°
	16–30	+20°	65°	86°		16–31	−32°	40°	81°

When the temperature is below zero, the distance is measured downward from the heavy horizontal line, as in the figure, page 149.

7. Average monthly rainfall at San Francisco, Valparaiso, Chili, and Quebec:

	(a) San Francisco	(b) Valparaiso	(c) Quebec
January . .	5.5 (inches)	0.2 (inches)	3.2 (inches)
February . .	4.5	0.3	6.4
March . . .	3.5	1	4.4
April . . .	2.5	2	6.6
May . . .	1.5	3	5.1
June . . .	0.5	4.2	2.5
July . . .	0.2	2.9	1.4
August . . .	0.2	1.8	1.8
September .	0.2	1.8	1.8
October . .	1.5	0.5	0.5
November . .	3.5	0.4	0.4
December . .	5.5	0.2	0.2

Use 10 small vertical spaces for 1 inch of rainfall and five horizontal spaces for 1 month.

8. After making the graphs in exercise 6, each on a separate sheet, put all three on the same sheet, using a different colored ink or pencil for each. In this way the relative average temperatures are simultaneously pictured.

9. After graphing (*a*), (*b*), and (*c*) of example 7, each on a separate sheet, combine all three, using different colors, as in 8.

10. Observe the weather reports in a daily paper and make a graph representing the hourly change of temperature for twenty-four hours.

GRAPHIC REPRESENTATION OF MOTION

105. A useful picture of the distance traversed by a moving body can be made by a graph similar to the preceding.

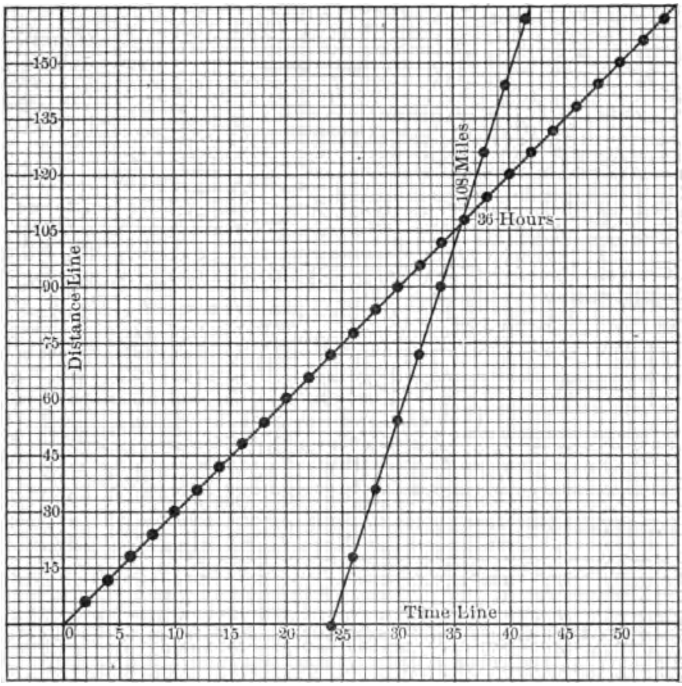
E.g. suppose a man is walking 3 miles per hour. We mark units of time from the starting point to the right along the horizontal reference line, and indicate miles traveled by the number of units measured vertically upward from this line. (See the figure on the opposite page.)

In the figure each horizontal space represents 1 hour, and each vertical space 3 miles. Then in 1 hour he goes 3 miles; in 5 hours, 15 miles; in 10 hours, 30 miles; etc. The dots representing the distances are found to lie on a straight line.

The graph shows at a glance the answers to such questions as: How many miles does he travel in 4 hours? in 13 hours? How long does it take him to go 18 miles? 23 miles?

Again, suppose 24 hours later a second man starts out on a bicycle to overtake the first man, and travels 9 miles an hour. The line drawn from the 24-hour point shows the distance the wheelman travels in any number of hours counting from his time of starting. The points marked in this line show how far he has gone in 1, 2, 3, 4, 5, 6 hours, etc., namely, 9, 18, 27, 36, 45, 54, etc.

The point where these two lines intersect shows in how many hours after starting the pedestrian is overtaken and also how far he has gone.



In like manner solve the following by means of graphs, and in each case suggest other questions which may be answered from the graph:

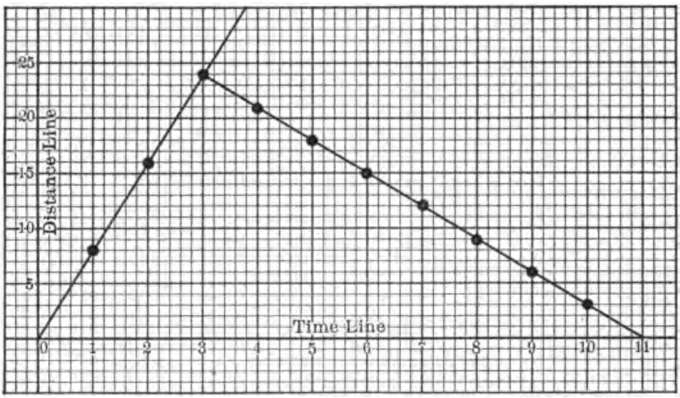
1. In a century bicycle race A averages 17 miles per hour; B, who starts 20 minutes later, averages 19 miles per hour. In how many hours will B overtake A? Who will win the race and where will the loser be when the winner finishes?

2. A starts for a town 12 miles distant, walking 3 miles per hour. $1\frac{1}{2}$ hours later B starts for the same place, driving $7\frac{1}{2}$ miles per hour. When does B overtake A? Where is A when B reaches town?

3. In a mile race A runs 6 yards per second and B 5 yards per second. B has a start of 250 yards. Who will win the race? How far in the lead is the winner at the finish?

Let 1 vertical small space represent 20 yards, and 1 horizontal space represent 5 seconds.

106. Illustrative Problem. A man rides a bicycle into the country at the rate of 8 miles per hour. After riding a certain distance the wheel breaks down and he walks back at the rate of 3 miles per hour. How far does he go if he reaches home 11 hours after starting?



In this graph each large horizontal space represents 1 hour, and each small vertical space represents 1 mile. The problem is solved as follows:

(1) Construct the line representing the outward journey at the rate of 8 miles per hour, extending this line indefinitely.

(2) Beginning at the point corresponding to 11 hours, find the points representing his position at each preceeding hour. The line connecting these points represents the homeward journey at the rate of 3 miles an hour. Extend this line until it meets the first line. The point where the lines meet represents 3 hours and 24 miles, which is the answer required in the problem.

PROBLEMS

Solve the following problems by means of graphs. In each case prepare a list of questions which may be answered from the graph.

1. A man rows 18 miles per hour down a river and 2 miles per hour returning. How far down the river can he go if he wishes to return in 10 hours?

2. A man goes from Chicago to Milwaukee on a train running $42\frac{1}{2}$ miles per hour, and returns immediately on a steamer going 17 miles per hour. Find the distance, if the round trip requires 7 hours.

3. A pleasure trip from New York to Atlanta by steamer and return by rail occupied 77 hours. Find the distance, if the rate going was 16 miles per hour and returning 40 miles per hour.

Let one small horizontal space represent one hour and one small vertical space 16 miles.

4. *A* invests \$1000 at 5% and *B* invests \$5000 at 4%. In how many years will the *amount* (principal and interest) of *A*'s investment equal the *interest* on *B*'s investment?

Let one large horizontal space represent one year and one small vertical space \$50. Then the line representing *A*'s *amount* starts at the point marked \$1000, and rises one small vertical space each year. The line representing *B*'s *interest* starts at the zero point and rises four small vertical spaces each year.

5. In how many years will the *interest* on \$6000 equal the *amount* on \$2000 if both are invested at 5%?

6. *A* invests \$500 at 6% and *B* invests \$1000 at 5%. In how many years will *A*'s interest differ by \$300 from *B*'s?

Let one small vertical space represent \$20. In this case both lines start from the zero point. Find the point on one line which is three large spaces *vertically* above the corresponding point on the other line.

7. Construct a graph representing the relation between the Fahrenheit and Centigrade thermometer readings.

Let the line at the bottom of the sheet and the vertical line four large spaces from the left margin be the reference lines. Let one small horizontal space represent a degree C, and one small vertical space a degree F.

From $F = 32 + \frac{9}{5}C$ (page 129), we find that if $C = 0°$ $F = 32°$, if $C = 10°$ $F = 50°$, if $C = 20°$ $F = 68°$, if $C = 30°$ $F = 86°$, etc. Mark the point representing each pair of readings, and draw the straight line connecting these points. This is the required graph.

8. From this graph read the answers to the following questions:

Find C when $F = 41°$, $F = 59°$, $F = 79°$, $F = 14°$.

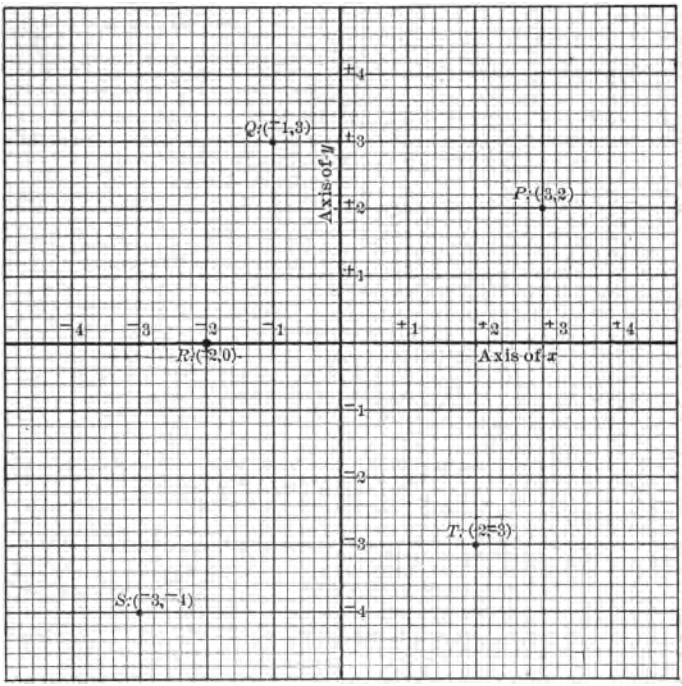
Find F when $C = 35°$, $C = 40°$, $C = -5°$, $C = -15°$.

From the graph it is possible to find any Fahrenheit reading when the corresponding Centigrade is given, and also to find any Centigrade reading when the corresponding Fahrenheit is given. Thus the graph shows to the eye all the information contained in the equation $F = 32 + \frac{9}{5}C$. In what follows we shall consider in detail how to represent equations in this manner.

GRAPHIC REPRESENTATION OF EQUATIONS

107. In all the graphs thus far constructed two lines at right angles to each other have been used as reference lines. These lines are called **axes**. The location of a point in the plane of such a pair of axes is completely described by giving its distance and direction from each of the axes. The direction to the right of the vertical axis is denoted by a positive sign, and to the left, by a negative sign; while direction upward from the horizontal axis is positive, and downward, negative.

The horizontal line is usually called the **x-axis** and the vertical line the **y-axis**. The perpendicular distance of any point P from the y-axis is called the **abscissa** of the point, and its distance from the x-axis is called its **ordinate**. The abscissa and ordinate of a point are together called its **coördinates**.



E.g. the abscissa of point P in the above figure is 3 and its ordinate 2, or we may say the coördinates of P are 3 and 2, and indicate it thus: $P:(3, 2)$, writing the abscissa first. In like manner for the other points we write $Q:(-1, 3)$, $R:(-2, 0)$, $S:(-3, -4)$, and $T:(2, -3)$. (For convenience *upper* signs are used in the figure.)

We see that in this manner every point in the plane corresponds to a pair of numbers and that every pair of numbers corresponds to a point. This scheme of locating points by two reference lines is already familiar to the pupil in geography, where cities are located by latitude and longitude, that is, by degrees north or south of the equator and east or west of the meridian of Greenwich.

EXERCISES

1. With any convenient scale, locate the following points: (2, 6), (– 3, 5), (0, 1), (1, 0), (0, 0), (0, –1), (0, – 5), (– 5, 0), ($2\frac{1}{2}$, $5\frac{1}{3}$), (–4, – 8), (3, – 10), (– 10, 3).

2. Calling north +, south –, east +, west –, so that (– 5°, 8°) means the point on a map whose longitude is 5° west and whose latitude is 8° north, verify the following on a map: New York (– 73° 57′, 40° 48′), Chicago (–87° 35′, 41° 51′), Peking (116° 30′, 39° 50′), Sydney, New South Wales (151° 15′, –33° 51′), Santiago, Chili (– 70° 39′, – 33° 25′).

3. On a map of South America give approximately the location of the following cities, using the notation of the preceding exercise: Caracas, Rio de Janeiro, Bogota, Valparaiso, Lima, and Panama.

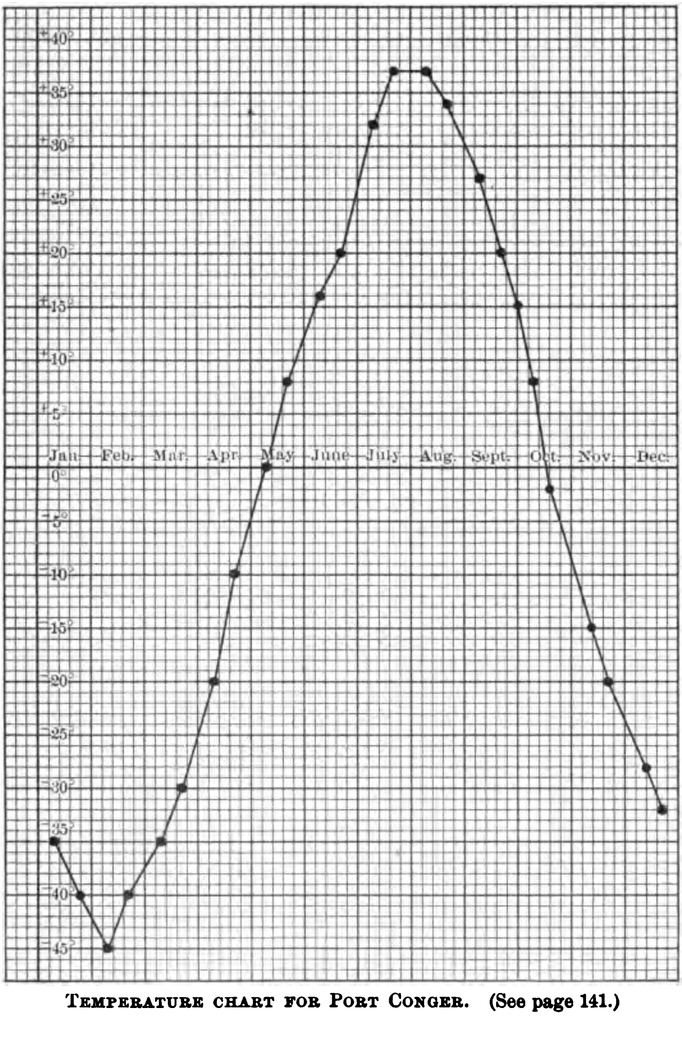
4. On a map of Africa locate in similar manner the cities or islands situated at the following points: (– 5° 40′, – 16°), (– 14° 30′, – 8°), (30° 18′, 30° 1′), (– 5° 40′, 36°), (40° 40′, – 15°), (18° 20′, – 33° 55′).

5. Locate the following series of points and then see if a straight line can be drawn through them: (0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (– 1, – 1), (– 2, – 2), (– 3, – 3). Name still other points lying on the same line.

6. Locate the following and connect them by a line: (1, 0), (1, 2), (1, 3), (1, 4), (1, 5), (1, – 2), (1, – 3), (1, – 4), (1, – 5). Name other points in this line.

7. Draw the line every one of whose points has its horizontal distance –2, also the line every one of whose points has its vertical distance + 3.

8. Locate the following points and see if a straight line can be passed through them: (1, 0), (0, 1), (2, – 1), (3, – 2), (4, – 3), (–1, + 2), (– 2, 3), (– 3, 4), (–4, 5), ($\frac{1}{2}$, $\frac{1}{2}$), ($\frac{1}{4}$, $\frac{3}{4}$), ($\frac{2}{3}$, $\frac{1}{3}$). Can you name other points on this line?



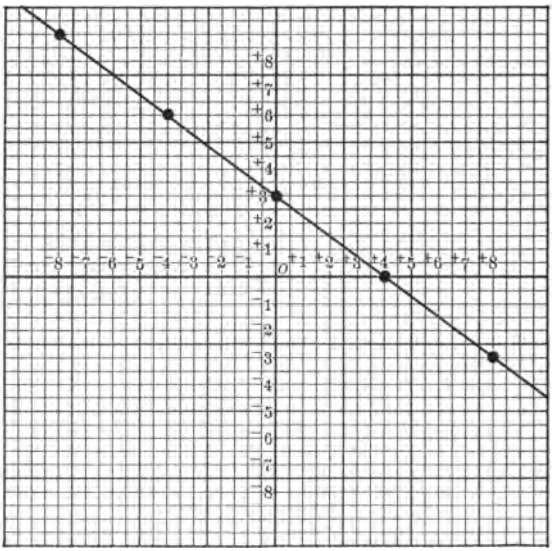
TEMPERATURE CHART FOR PORT CONGER. (See page 141.)

108. In the preceding exercises, in certain cases, a series of points has been found to **lie on a straight line** as in examples 6, 7, and 8. Evidently this could not happen unless the points were located according to some definite scheme or law.

Illustrative Problem. Locate a series of points whose coordinates are values of x and y which satisfy: $3x + 4y = 12$.

We see that $x = 0$, $y = 3$, also $x = 4$, $y = 0$ are pairs of such values. Evidently as many pairs of values as we please may be found by giving any value to x and then solving the equation to find the corresponding value of y. A table may thus be constructed as follows:

Let	$x = 0,\ 4,\ 8,\ 12,\ -4,\ -8$, etc.
Then	$y = 3,\ 0,\ -3,\ -6,\ 6,\ 9$, etc.



These pairs of values for x and y correspond to the points as plotted in the figure, and they are found to lie on a straight line. This line is called the **graph of the equation.**

Let the student find other pairs of numbers which satisfy this equation and see if the corresponding points lie on this line. Also find the numbers which correspond to any chosen point on this line and see whether they satisfy the equation.

109. Since an equation like $3x+4y=12$ is satisfied by indefinitely many pairs of values of x and y, it is common to call the unknowns in such an equation **variables**. Indeed if a point be thought of as moving along the graph of this equation the values of x and y corresponding to the moving point continually vary, *but always so that* $3x+4y=12$.

110. **Definitions.** An equation is said to be of the **first degree in x and y** if it contains each of these letters in such a way that neither x nor y is multiplied by itself or by the other.

E.g. $13x-5y=14$ is of the first degree, while $2xy-x=5$ and $3x-5y^2=13$ are not of the first degree in x and y.

Every equation of the first degree in two variables has for its graph a straight line; hence such an equation is commonly called a **linear equation.**

111. To graph an equation of the first degree it is only necessary to find two points on the graph and draw a straight line through them.

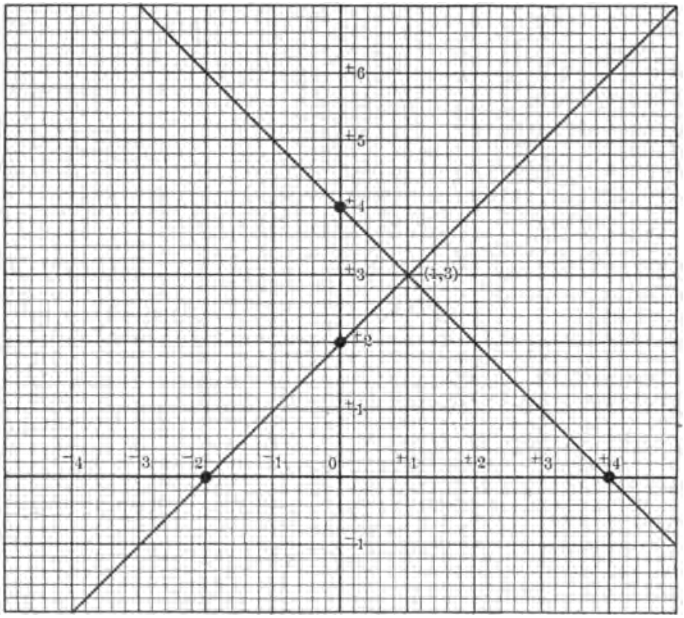
E.g. In graphing the equation $x-y=5$, we choose $x=0$ and find $y=-5$, and choose $y=0$ and find $x=5$ and plot the points $(0, -5)$ and $(5, 0)$. The line through these points is the one required.

EXERCISES

Construct the graph for each of the following equations:

1. $3x+2y=1$.
2. $5x-3y=-3$.
3. $7x+10y=2$.
4. $x+2y=0$.
5. $5-2y=12$.
6. $3x+5y=-15$.
7. $2x-y=0$.
8. $3x-4y=7$.
9. $3x-4y=-7$.
10. $3x-4y=-12$.
11. $7y=9x-63$.
12. $x=5y+3$.

112. Illustrative Problem. Graph on the same axes the two equations $x+y=4$ and $y-x=2$.



Solution. The two graphs are found to intersect in the point (1, 3). Since the point lies on both lines, its coördinates should satisfy both equations, as indeed they do. Since these lines have only one point in common, there is no other pair of numbers which, when substituted for the variables x and y, can satisfy both equations.

Hence $x=1$, $y=3$, which is written (1, 3), is called the **solution of this pair of equations.**

113. Definition. These two equations are called **independent** because their graphs are distinct. They are called **simultaneous** because there is at least one pair of values of x and y which satisfy both.

114. Since two straight lines intersect in but one point, it follows that two linear equations which are independent and simultaneous have one and only one solution.

EXERCISES

Graph the following and thus find the solution of each pair of equations:

1. $\begin{cases} 2x - 3y = 25, \\ x + y = 5. \end{cases}$

2. $\begin{cases} 5x + 6y = 7, \\ 2x - y = -4. \end{cases}$

3. $\begin{cases} 5x + 3y = 1, \\ 2x + y = -4. \end{cases}$

4. $\begin{cases} 6x + 8y = 16, \\ 2x - 3y = 11. \end{cases}$

5. $\begin{cases} 3x - 4y = 1, \\ 2x - 7y = 5. \end{cases}$

6. $\begin{cases} y + 3x = 7, \\ 2y + x = -6. \end{cases}$

7. $\begin{cases} x - 2y = 2, \\ 2x - y = -2. \end{cases}$

8. $\begin{cases} 5x - 7y = 21, \\ x - 4y = -1. \end{cases}$

9. $\begin{cases} 5x + 2y = 8, \\ 2x - 3y = -12. \end{cases}$

10. $\begin{cases} x + 3y = -6, \\ 2x - 4y = -12. \end{cases}$

SOLUTION OF SIMULTANEOUS EQUATIONS BY SUBSTITUTION

115. A pair of linear equations may be solved without constructing their graphs.

Illustrative Problem. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Solution. Let x = the greater number and y the smaller.

Then
$$\begin{cases} x + y = 35, & (1) \\ x - y = 5. & (2) \end{cases}$$
and

From (1) by S, $y = 35 - x$. (3)

Substituting $35 - x$ for y in (2), $x - (35 - x) = 5$. (4)

Solving, $x = 20$. (5)

Substituting $x = 20$ in (1), $y = 15$. (6)

116. In most problems solved heretofore there have been two or more unknown numbers. In forming the equations the object has been to express all but one of the unknowns in terms of that one. In the case of two unknowns this is now done more systematically as follows:

(*a*) State two equations involving the two unknowns, as (1) and (2) above.

(*b*) Solve one of these equations for one unknown in terms of the other as in (3) above.

(*c*) Substitute in the other equation the value of this unknown as thus expressed, obtaining as in (4) one equation in one unknown.

This equation (4) is the same as equation (1) on page 111, where this problem was solved by means of one unknown. The solution given here amounts to a *formal tabulation* of the steps there taken in obtaining the equation, $g-(35-g)=5$.

Since equation (4) contains but one of the unknowns, the other is said to be **eliminated.**

The process here used is called **elimination by substitution.**

Illustrative Problem. Solve the equations:

$$\begin{cases} 2x+3y=13. & (1) \\ 5x-6y=-8. & (2) \end{cases}$$

From (1) by S and D, $$y=\frac{13-2x}{3}. \quad (3)$$

Substituting in (2) $$5x-\frac{6(13-2x)}{3}=-8. \quad (4)$$

By F, V, $$5x-2(13-2x)=-8. \quad (5)$$

By IV, VII, $$5x-26+4x=-8. \quad (6)$$

By I, A, $$9x=18. \quad (7)$$

By D, $$x=2. \quad (8)$$

From (3), $$y=\frac{13-2\cdot 2}{3}=3. \quad (9)$$

Verify this by drawing the graphs and also by substituting these values of x and y in (1) and (2).

EXERCISES

Solve the following pairs of linear equations by eliminating one of the variables by substitution, and check by substituting the results in the original equations.

1. $\begin{cases} x + y = 4, \\ x - y = 10. \end{cases}$

2. $\begin{cases} x - y = -3, \\ x + 4y = 12. \end{cases}$

3. $\begin{cases} 2x + 3y = 5, \\ 7x - 5y = 33. \end{cases}$

4. $\begin{cases} 3x - 4y = 8, \\ 4x + 3y = -6. \end{cases}$

5. $\begin{cases} 2x - 4y = 8, \\ 3x + 2y = 4. \end{cases}$

6. $\begin{cases} x + 2y = 4, \\ 2x + y = -1. \end{cases}$

7. $\begin{cases} 3x - 4y = 8, \\ 2x + 3y = 11. \end{cases}$

8. $\begin{cases} 5x + 9y = 19, \\ 3x - y = 5. \end{cases}$

9. $\begin{cases} 4y - 2x = 3, \\ 2y + 5x = 6. \end{cases}$

10. $\begin{cases} 3x - 7y = -11, \\ 2x + y = 4. \end{cases}$

11. $\begin{cases} 5x - 3y = 4 - 2x + 7y, \\ 5y + x = 7. \end{cases}$

12. $\begin{cases} 3y + 5x = 12 + 2x, \\ 17x - y = 4y - 20. \end{cases}$

13. $\begin{cases} 6y - x = 7 + 4y, \\ 5x + 8y = 1. \end{cases}$

14. $\begin{cases} 6 + x + y = 2x - 1, \\ 3y + x = 6y + 9. \end{cases}$

15. $\begin{cases} x - y = 37, \\ 2x + 3y = 314 + 13y. \end{cases}$

16. $\begin{cases} 2x + 3y = y + 6, \\ x + 2y = 4y + 3. \end{cases}$

17. $\begin{cases} y + 5x = 2x + 5, \\ 2y - 3x = 19. \end{cases}$

18. $\begin{cases} 5x + 3y = 0, \\ 2x + y = 1. \end{cases}$

19. $\begin{cases} 2x + 3y = 6x - 1, \\ 3x - 2y = 3. \end{cases}$

20. $\begin{cases} 5x - 3y = 0, \\ 2x + 2 - 6y = 2 - x. \end{cases}$

21. $\begin{cases} 6x + 2y = 23, \\ 10x - 5y = 21. \end{cases}$

22. $\begin{cases} 3x - 7y = 15, \\ 5x + 4y = 11. \end{cases}$

Solve 21 and 22 by means of graphs and also by elimination by substitution. Notice that the latter method gives a more accurate solution than can be obtained from the graph.

SOLUTION OF SIMULTANEOUS EQUATIONS BY ADDITION OR SUBTRACTION

117. Illustrative Examples. Solve

$$\begin{cases} x + 2y = 7, & (1) \\ 3x - 2y = 5. & (2) \end{cases}$$

Adding the members of these equations, $+2y$ and $-2y$ cancel.

Hence, $$4x = 12, \qquad (3)$$

$$x = 3. \qquad (4)$$

Substituting in (1), $$3 + 2y = 7, \qquad (5)$$

$$y = 2. \qquad (6)$$

Verify this by drawing the graphs, and also by substituting $x = 3$, $y = 2$, in (1) and (2).

If one of the variables does not already have the same coefficient in both equations, the solution may be obtained as follows:

Solve the equations

$$\begin{cases} 7x + 3y = 4 - y + 4x, & (1) \\ 3x - y = 4y - 2 - x. & (2) \end{cases}$$

From (1) by A, S, $$3x + 4y = 4. \qquad (3)$$

From (2) by A, S, $$4x - 5y = -2. \qquad (4)$$

From (3) by M, $$12x + 16y = 16. \qquad (5)$$

From (4) by M, $$12x - 15y = -6. \qquad (6)$$

Subtracting (6) from (5), $$31y = 22. \qquad (7)$$

From (7) by D, $$y = \tfrac{22}{31}. \qquad (8)$$

Substituting in (3), $$x = \tfrac{12}{31}. \qquad (9)$$

Hence, the solution is $\frac{12}{31}$, $\frac{22}{31}$.

118. The process used in the solution just given is called **elimination by addition or subtraction.**

This method is usually simpler than elimination by substitution, since the latter frequently involves fractions.

EXERCISES

Solve the following pairs of equations by addition or subtraction. Substitute the results in the given equations in each case to test the accuracy of the solution.

1. $\begin{cases} 2x + 3y = 22, \\ x - y = 1. \end{cases}$

2. $\begin{cases} 5x - 2y = 21, \\ x - y = 6. \end{cases}$

3. $\begin{cases} 6x + 30 = 8y, \\ 3y + 17 = 2 - 3x. \end{cases}$

4. $\begin{cases} 8x - 4y = 12x, \\ 4x + 2y = 3 + 4y. \end{cases}$

5. $\begin{cases} x + 6y = 2x - 16, \\ 3x - 2y = 24. \end{cases}$

6. $\begin{cases} 5x + 10y = -7, \\ 2x + 5y = -2. \end{cases}$

7. $\begin{cases} 5x + 3y = -2, \\ 3x + 2y = -1. \end{cases}$

8. $\begin{cases} 3a + 7b = 7, \\ 5a + 3b = 29. \end{cases}$

9. $\begin{cases} r = 3s - 19, \\ s = 3r - 23. \end{cases}$

10. $\begin{cases} 2p = 5q - 16, \\ 7q = -3p + 5. \end{cases}$

Solve the following by either process of elimination:

1. $\begin{cases} 7m = 2n - 3, \\ 19n = 6m + 89. \end{cases}$

2. $\begin{cases} 6c + 15d = -6, \\ 21d - 8c = -74. \end{cases}$

3. $\begin{cases} 2x - 3y = 4, \\ 2y - 3x = -21. \end{cases}$

4. $\begin{cases} u + v = 27, \\ \frac{2}{3}v = 19 - \frac{3}{4}u. \end{cases}$

5. $\begin{cases} 7a = 1 + 10y, \\ 16y = 10a - 1. \end{cases}$

6. $\begin{cases} 15k = 10 - 20l, \\ 25k - 30l = 80. \end{cases}$

7. $\begin{cases} 28x + 14y = 23, \\ 14x - 14y = 1. \end{cases}$

8. $\begin{cases} 5x + 2y = x + 18, \\ 2x + 3y = 3x + 27. \end{cases}$

9. $\begin{cases} 7y - x = x - 17, \\ 2y + 3x = 38. \end{cases}$

10. $\begin{cases} 6x + 2y = -2, \\ x - 4y = -35. \end{cases}$

11. $\begin{cases} 3x - y = 2x - 1, \\ 12x + y = 14. \end{cases}$

12. $\begin{cases} 2x + 3y = 5, \\ 6x + 14y = 0. \end{cases}$

13. $\begin{cases} 4x + 3y = 5, \\ 7x - 2y = 74. \end{cases}$

14. $\begin{cases} 6y + 2x = 11, \\ 3y + 12x = 18. \end{cases}$

15. $\begin{cases} 4y + 9x = -5, \\ x + y = -5. \end{cases}$

16. $\begin{cases} 7x - 4y = 3, \\ 5x + 8y = 6. \end{cases}$

17. $\begin{cases} 12y - 10x = -6, \\ 7y + x = 99. \end{cases}$

18. $\begin{cases} 7x - 3y = -7, \\ 5y - 9x = 1. \end{cases}$

19. $\begin{cases} 7x + 4y = 3, \\ 2x + 3y = 25. \end{cases}$

20. $\begin{cases} 34x + 70y = 4, \\ 5x - 8y = -36. \end{cases}$

119. The equations thus far given have for the most part been written in a standard form, $ax + by = c$, in which all the terms containing x are collected, likewise those containing y, and those which contain neither variable. When the equations are not given in this form, they should be so reduced, as in the second solution on page 156, before applying any method of elimination, and also before solving by means of graphs.

EXERCISES

After reducing each of the following pairs of equations to the standard form, solve by graphing, or by means of either process of elimination, as seems best available.

1. $\begin{cases} x - 14 = 7y, \\ 6y + 1 = x. \end{cases}$

2. $\begin{cases} 16x - 3y = 7x, \\ 4y = 7x + 5. \end{cases}$

3. $\begin{cases} r + 1 = -4s, \\ 2s = 13 - 5r. \end{cases}$

4. $\begin{cases} m = \dfrac{1 - 8n}{5}, \\ 3m + 5n = 1. \end{cases}$

5. $\begin{cases} m - n = 16, \\ 3m = 8 - 2n. \end{cases}$

6. $\begin{cases} \dfrac{7x - 15}{3} = y, \\ 2x - y = 3. \end{cases}$

7. $\begin{cases} \dfrac{x - 3}{5y} = -2, \\ x + 7y = 6. \end{cases}$

8. $\begin{cases} 3x - 5 = -y, \\ 8y + 76 = 5x. \end{cases}$

9. $\begin{cases} a + 4b = 14, \\ 3a - b = 14. \end{cases}$

10. $\begin{cases} \dfrac{x + y}{2} + \dfrac{x - y}{2} = 10, \\ 2x - y = 16. \end{cases}$

11. $\begin{cases} \dfrac{x - y}{5} + \dfrac{x + y}{3} = 8, \\ \dfrac{2x - y}{2} - \dfrac{3y - x}{4} = 12\frac{1}{2}. \end{cases}$

12. $\begin{cases} \dfrac{7m + 8}{5} - \dfrac{7n - 1}{4} = -2, \\ \dfrac{2m - 4}{2} + \dfrac{n - 1}{3} = -\frac{1}{3}. \end{cases}$

13. $\begin{cases} \dfrac{x - 3}{4} + \dfrac{y + 8}{5} = 2, \\ \dfrac{x + 7}{2} + \dfrac{2y - 4}{7} = 5. \end{cases}$

14. $\begin{cases} \dfrac{8a - 3}{9} + \dfrac{5b - 2}{3} = 13, \\ \dfrac{2a + 7}{5} - \dfrac{3b + 10}{10} = -3\frac{4}{5}. \end{cases}$

15. $\begin{cases} \dfrac{7y - 4}{5} + \dfrac{2x - 3}{2} = 7, \\ \dfrac{6x - 3}{5} + \dfrac{2y + 1}{5} = 7. \end{cases}$

16. $\begin{cases} \dfrac{3y + 7}{2} - \dfrac{5x - 7}{3} = 10, \\ \dfrac{2x - 4}{3} - \dfrac{2y - 1}{4} = -2\frac{1}{4}. \end{cases}$

17. $\begin{cases} \dfrac{5 + 3p}{7} - \dfrac{5q - 2}{4} = -2, \\ 6p + 8q = 108. \end{cases}$

18. $\begin{cases} 3x - 2y = 4, \\ \dfrac{2x - 1}{5} - \dfrac{7y - 4}{3} = -19. \end{cases}$

19. $\begin{cases} 5x + 7y = 89\frac{1}{2}, \\ \dfrac{2x - 4}{3} + \dfrac{6y - 1}{5} = 13\frac{1}{3}. \end{cases}$

20. $\begin{cases} 32x - 9y = 299, \\ \dfrac{2x - 5}{7} - \dfrac{3y - 1}{2} = -16. \end{cases}$

PROBLEMS

Solve the following problems, using two variables in each case.

1. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.

2. Find two numbers such that 7 times the first plus four times the second equals 37; while 3 times the first plus 9 times the second equals 45.

3. A certain sum of money was invested at 5% interest and another sum at 6%, the two investments yielding $980 per annum. If the first sum had been invested at 6% and the second at 5%, the annual income would be $1000. Find each sum invested.

4. The combined weight of 3 cubic centimeters of platinum and 50 cubic centimeters of poplar is 84 grams, and the weight of 1 cubic centimeter of platinum and 150 cubic centimeters of poplar is 80 grams. Find the weight of 1 cubic centimeter of each.

5. The combined distance from the sun to Jupiter and from the sun to Saturn is 1369 million miles. Saturn is 403 million miles farther from the sun than Jupiter. Find the distance from the sun to each planet.

6. The sum of the distances from the sun to the fixed stars Altair and Capella is 45.3 light-years. Twice the distance of Altair plus 3 times that of Capella is 119.6 light-years. Find the distance from each star to the sun.

7. Find two numbers such that 7 times the first plus 9 times the second equals 116, and 8 times the first minus 4 times the second equals 4.

8. The sum of two numbers is 108. 8 times one of the numbers is 9 greater than the other number. Find the numbers.

9. Two investments of $24,000 and $16,000 respectively yield a combined income of $840. The rate of interest on the larger investment is 1% greater than that on the other. Find the two rates of interest.

10. A father is twice as old as his son. Twenty years ago the father was six times as old as his son. How old is each now?

11. If the length of a rectangle is increased by 3 feet and its width decreased by 1 foot, its area is increased by 3 square feet. If the length is increased by 4 feet and the width decreased by 2 feet, the area is decreased by 3 square feet. What are the dimensions of the rectangle?

Let $l =$ the original length and w the width,

then $$(l+3)(w-1) = lw + 3, \quad (1)$$

and $$(l+4)(w-2) = lw - 3. \quad (2)$$

From (1) by XIII, $$lw + 3w - l - 3 = lw + 3. \quad (3)$$

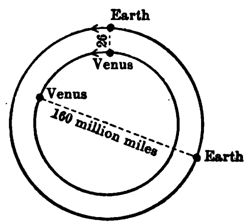
From (2) by XIII, $$lw + 4w - 2l - 8 = lw - 3. \quad (4)$$

From (3) and (4) $$3w - l = 6, \quad (5)$$

and $$4w - 2l = 5. \quad (6)$$

(5) and (6) may now be solved in the ordinary manner.

12. The greatest distance from the earth to Venus is 160 million miles and the shortest distance is 26 million miles. How far from the sun are Venus and the earth, assuming that they move around the sun in concentric circles with the sun at the center?



13. Two weights, 35 and 40 pounds respectively, balance when resting on a beam at certain unknown distances from the fulcrum. If 15 pounds is added to the 35-lb. weight, the 40-lb. weight must be moved 2 feet farther from the fulcrum in order to maintain the balance. What

was the original distance from the fulcrum to each of the weights?

If the distances from the fulcrum to the weights are d_1 and d_2 respectively, then by the formula, $w_1d_1 = w_2d_2$, given on page 121, we have $35\,d_1 = 40\,d_2$ and $50\,d_1 = 40\,(d_2 + 2)$.

14. A steamer on the Mississippi makes 6 miles per hour going against the current and $19\frac{1}{2}$ miles per hour going with the current. What is the rate of the current and at what rate can the steamer go in still water?

15. A man starts at 7 A.M. for a walk in the country. At 10 A.M. another man starts on horseback to overtake the pedestrian, which he does at 1 P.M. If the rate of the horseman had been two miles per hour less, he would have overtaken the pedestrian at 4 P.M. At what rate does each travel?

16. A camping party sends a messenger with mail to the nearest post office at 5 A.M. At 8 A.M. another messenger is sent out to overtake the first, which he does in $2\frac{1}{4}$ hours. If the second messenger travels 5 miles per hour faster than the first, what is the rate of each?

17. There are two numbers such that 3 times the greater is 18 times their difference, and 4 times the smaller is 4 less than twice the sum of the two. What are the numbers?

18. Roy is three times as old as Fred was 8 years ago. Five years from now Roy will be 16 years less than twice as old as Fred. How old is each now?

19. A picture is 3 inches longer than it is wide. The frame 4 inches wide has an area of 360 square inches. What are the dimensions of the picture?

20. The difference between 2 sides of a rectangular wheat field is 30 rods. A farmer cuts a strip 5 rods wide around the field, and finds the area of this strip to be $7\frac{1}{2}$ acres. What are the dimensions of the field?

21. The sum of the length and width of a certain field is 260 rods. If 20 rods are added to the length and 10 rods to the width, the area will be increased by 3800 square rods. What are the dimensions of the field?

22. In a number consisting of two digits the sum of the digits is 12. If the order of the digits is reversed the number is decreased by 36. What is the number?

23. A bird attempting to fly against the wind is blown backward at the rate of $7\frac{1}{2}$ miles per hour. Flying with a wind $\frac{1}{4}$ as strong, the bird makes 48 miles an hour. Find the rate of the wind and the rate at which the bird can fly in calm weather.

24. There is a number whose two digits differ by 2. If the digit in units' place is multiplied by 3 and the digit in tens' place is multiplied by 2, the number is increased by 44. Find the number, the tens' digit being the larger.

25. In a number consisting of two digits one digit is equal to twice their difference. If the order of the digits is reversed, the number is increased by 18. Find the number, the units' digit being the larger.

26. If the length of a rectangle is doubled and 8 inches added to the width, the area of the resulting rectangle is 180 square inches greater than twice the original area. If the length and width of the rectangle differ by 10, what are its dimensions?

27. The Centigrade reading at the boiling point of alcohol is 96° lower than the Fahrenheit reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

Use C and F as the unknowns. Then one of the equations is the formula connecting Fahrenheit and Centigrade readings obtained on page 129, and the other is $C + 96 = F$.

28. The Centigrade reading at the boiling point of mercury is 312° lower than the Fahrenheit reading. Find both the Fahrenheit and the Centigrade reading at this temperature.

29. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

30. A man rowing against a tidal current drifts back $2\frac{1}{4}$ miles per hour. Rowing with this current, he can make $12\frac{1}{8}$ miles per hour. How fast does he row in still water and how swift is the current?

31. Flying against a wind a bird makes 28 miles per hour, and flying with a wind whose velocity is $2\frac{3}{5}$ times as great, the bird makes 46 miles per hour. What is the velocity of the wind and at what rate does the bird fly in calm weather?

32. A freight train leaves Chicago for St. Paul at 11 A.M. At 3 and 5 P.M. respectively of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 P.M. the same day, and the other, which runs 10 miles per hour slower, at 3 A.M. the next day. What is the speed of each?

33. Two boys, A and B, having a 30-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when B is 6 feet and A 5 feet from the fulcrum. If B places the 30-lb. weight on the board beside him, they balance when B is 4 and A 5 feet from the fulcrum. How heavy is each boy?

34. C is $6\frac{1}{2}$ feet from the point of support and balances D, who is at an unknown distance from this point. C places a 33-lb. weight beside himself on the board and when $4\frac{2}{3}$ feet from the fulcrum, balances D, who remains at the same point as before. D's weight is 84 pounds. What is C's weight and how far is D from the fulcrum?

35. E weighs 95 pounds and F 110 pounds. They balance at certain unknown distances from the fulcrum. E then takes

a 30-lb. weight on the board, which compels F to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

36. A fast freight leaves New York for Chicago at 8 A.M. At 4 P.M. the same day an express train leaves New York for Chicago and passes the freight 12 hours later. Another express leaving New York at 6 P.M. of the same day overtakes the freight 10 hours after starting. Find the rate of each train if the second express goes 8 miles per hour faster than the first.

37. The Centigrade reading at the melting point of silver is 796° lower than the Fahrenheit reading. Find both Centigrade and Fahrenheit readings at this temperature.

38. The Fahrenheit reading at the melting point of gold is 992° higher than the Centigrade reading. Find both Centigrade and Fahrenheit readings at this temperature.

39. $10,000 and $8000 are invested at different rates of interest, yielding together an annual income of $820. If the first investment were $12,000 and the second $6000, the yearly income would be $840. Find the rates of interest.

40. In a switch yard a car weighing 50 tons and going at a certain rate strikes a standing car, whereupon both cars move off at the rate of 4 miles per hour. If the second car, moving at the same rate as the first before impact, were to strike the first car when standing still, they would move off at the rate of 2 miles per hour. How fast did the first car move before impact and what is the weight of the second car? (Set up the equations by means of the formula $w_1v_1 = w_1v_1' + w_2v_2$, obtained on page 127.)

41. 200 ccm. of white oak is fastened to 25 ccm. of steel, making a combination whose average density is 1.56. If 250 ccm. of oak is fastened to 20 ccm. of steel, the average density of the combination is 1.3. Find the density of white oak and also of steel.

SIMULTANEOUS EQUATIONS IN THREE VARIABLES

120. Illustrative Problem. Three men were discussing their ages and found that the sum of their ages was 90 years. If the age of the first were doubled and that of the second trebled, the aggregate of the three ages would then be 170. If the ages of the second and third were each doubled, the sum of the three would be 160. Find the age of each?

Solution. Let x, y, and z represent the number of years in their ages in the order named.

Then, $$x + y + z = 90, \quad (1)$$

$$2x + 3y + z = 170, \quad (2)$$

and $$x + 2y + 2z = 160. \quad (3)$$

Since by supposition x represents the same number in all three equations, and likewise y and z, if we subtract (1) from (2), we obtain a new equation from which z is eliminated.

I.e. $$x + 2y = 80. \quad (4)$$

Again, multiplying (2) by 2 and subtracting (3),

$$3x + 4y = 180. \quad (5)$$

(4) and (5) are two equations in the two variables x and y. Solving these by eliminating y, we find $x = 20$. (6)

Substituting $x = 20$ in (4), $y = 30$. (7)

Substituting x and y in (1), $z = 40$. (8)

Check by substituting the values of x, y, and z in all three given equations and also by showing that they satisfy the conditions of the problem.

The values of x, y, and z as thus found constitute the **solution of the given system** of equations.

Evidently x could have been eliminated first, using (1), (2) and (1), (3), giving a new set of two equations in y and z. Let the student find the solution in this manner.

Also find the solution by first eliminating y, using (1), (2) and (2), (3), getting two equations in x and z, from which the values of x and z can be found.

121. Definition. An equation is said to be of the **first degree in three variables** if no one of the variables is multiplied by itself or by one of the others (§ 110).

The fact that the solutions are found to be the same no matter in what order the equations are combined, indicates that *a system of three independent and simultaneous equations of the first degree in three variables has one and only one solution.*

As in the case of two equations, each should be first reduced to a standard form in which all the terms containing a given variable are collected and united and all fractions removed by *M*, Principle VIII.

EXERCISES

Solve the following systems of equations, and check the results by substituting the values found for each variable in the given equations:

1. $\begin{cases} 2x - y + z = 18, \\ x - 2y + 3z = 10, \\ 3x + y - 4z = 20. \end{cases}$

2. $\begin{cases} 5x - 3y + z = 15, \\ x + 3y - z = 3, \\ 2x - y + z = 8. \end{cases}$

3. $\begin{cases} 4x + 2y + z = 13, \\ x - y + z = 4, \\ x + 2y - z = 1. \end{cases}$

4. $\begin{cases} 6x + 4y - 4z = -4, \\ 4x - 2y + 8z = 0, \\ x + y + z = 4. \end{cases}$

5. $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 5, \\ x + y + z = 2. \end{cases}$

6. $\begin{cases} 2x - 8y + 3z = 2, \\ x - 4y + 5z = 1, \\ 3x - 10y - z = 5. \end{cases}$

7. $\begin{cases} x + y + z = 1, \\ x + 3y + 2z = 8, \\ 2x + 8y - 3z = 15. \end{cases}$

8. $\begin{cases} 2x - 3y + z = 5, \\ 3x + 2y - z = 5, \\ x + y + z = 3. \end{cases}$

9. $\begin{cases} x + y + z = 6, \\ 3x - 2y - z = 13, \\ 2x - y + 3z = 26. \end{cases}$

10. $\begin{cases} x + y + z = 6, \\ 4x - y - z = -1, \\ 2x + y - 3z = -6. \end{cases}$

PROBLEMS INVOLVING THREE VARIABLES

122. Illustrative Problem. A broker invested a total of \$15,000 in the street railway bonds of three cities, the first investment yielding 3 %, the second $3\frac{1}{2}$ %, and the third 4 %, thus securing an income of \$535 per year. If the second investment was one-half the sum of the other two, what was the amount of each?

Solution. Suppose x dollars were invested at 3 %, y dollars at $3\frac{1}{2}$ %, and z dollars at 4 %.

Then,	$x + y + z = 15000,$	(1)
	$.03\,x + .035\,y + .04\,z = 535,$	(2)
and	$x + z = 2\,y.$	(3)
From (3),	$x - 2\,y + z = 0.$	(4)
Subtract (4) from (1),	$3\,y = 15000,$	(5)
and	$y = 5000.$	(6)
From (1), by M,	$.035\,x + .035\,y + .035\,z = 525.$	(7)
Subtract (7) from (2),	$-.005\,x + .005\,z = 10.$	(8)
Divide (8) by .005,	$-x + z = 2000.$	(9)
Substitute (6) in (4),	$x + z = 10000.$	(10)
Add (9) and (10),	$2\,z = 12000.$	(11)
	$z = 6000.$	(12)
Substitute (6) and (12) in (1),	$x = 4000.$	(13)

Hence, \$4000, \$5000, and \$6000 were the sums invested.

Solve the following problems, using three unknowns:

1. The sum of three angles A, B, and C of a triangle is 180 degrees. $\frac{1}{3}$ of $A + \frac{1}{4}$ of $B + \frac{1}{5}$ of C is 48 degrees, while $\frac{1}{6}$ of $A + \frac{1}{8}$ of $B + \frac{1}{4}$ of C is 30 degrees. How many degrees in each angle?

2. The combined weight of 1 cubic foot each of compact limestone, granite, and marble is 535 pounds. 1 cubic foot of limestone, 2 of granite, and 3 of marble weigh together 1041 pounds, while 1 cubic foot of limestone and 1 of granite together weigh 195 pounds more than 1 cubic foot of marble. Find the weight per cubic foot of each kind of stone.

3. A number is composed of 3 digits whose sum is 7. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the order of the digits is reversed, the number is decreased by 99. What is the number?

4. The sum of the angles A, B, and C of a triangle is 180 degrees. If B is subtracted from C the remainder is $\frac{1}{8}$ of A, and when C is subtracted from twice A the remainder is 4 times B. How many degrees in each angle?

5. The sum of the three sides a, b, c of a certain triangle is 35, and twice a is 5 less than the sum of b and c, and twice c is 4 more than the sum of a and b. What is the length of each side?

6. The combined number of students at Harvard, Yale, and Columbia during the year 1905–1906 was 13,390. The number at Harvard minus the number at Columbia plus twice the number at Yale was 6893, and the number at Columbia plus 4 times the number at Yale minus twice the number at Harvard was 7258. What was the number at each university?

7. The total number of students at the universities of Illinois, Michigan, and Wisconsin during the year 1905–1906 was 12,216. Twice the number at Illinois plus 3 times that at Michigan plus 4 times that at Wisconsin was 36,145. If the number at Michigan is subtracted from the sum of the numbers at Illinois and Wisconsin, the remainder is 3074. Find the number at each university.

8. The total number attending schools in the United States in 1904–1905 was 17,953,844. If the number in secondary schools and colleges combined be subtracted from the number in elementary schools, the remainder is 15,924,656; while if twice the number in colleges be added to the number in elementary and secondary schools combined, the sum is 18,092,388. Find the number of students of each kind.

9. The combined foreign trade in 1905 at the three ports, London, Liverpool, and New York, was 3598 million dollars. If the amount at London is subtracted from the combined amounts at New York and Liverpool, the remainder is 988 million; and if the amount at New York is subtracted from the combined amounts at London and Liverpool, the remainder is 1384 million. Find the amount of foreign trade at each port.

In the following three examples find the number of seconds in each record and reduce the results to minutes and seconds.

10. If x is the number of seconds in the Eastern intercollegiate record for a mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} x+y+z=781.15, \\ -x+2y+z=519.35, \\ 2x-y+z=514.55. \end{cases}$$

11. If x is the number of seconds in the Eastern intercollegiate record for a half mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} 2x+3y+z=697.7, \\ 3x+2y+2z=809.8, \\ 2x-y+z=228.1. \end{cases}$$

12. If $x=$ number of seconds in the world's mile trotting record in 1806, $y=$ number of seconds in the world's record in 1885, and $z=$ number of seconds in the world's record in 1903, then

$$\begin{cases} x+y+z=426.25, \\ 2x+4y+6z=1584, \\ -x+y+2z=186.75. \end{cases}$$

REVIEW QUESTIONS

1. How may a point in a plane be located by reference to two fixed lines? What are these fixed lines called? What names are given to the distances from the point to the fixed lines? Why are negative numbers needed in order to locate all points in this manner?

2. Draw a pair of axes in a plane and locate the following points: (5,0), $(-2,0)$, (0,3), $(0,-1)$, (0,0).

3. State a problem involving motion and solve it by means of a graph.

4. How many pairs of numbers can be found which satisfy the equation $x-2y=6$? State five such pairs and plot the corresponding points. How are these points situated with respect to each other? What can you say of all points corresponding to pairs of numbers which satisfy this equation? What is meant by the graph of an equation?

5. How many pairs of numbers will simultaneously satisfy the two equations $3x+2y=7$ and $x+y=3$? Show by means of a graph that your answer is correct.

6. Describe elimination by the process of substitution; also by the process of addition or subtraction. Under what conditions is one or the other of these methods preferable?

7. Why is the solution by elimination in some cases preferable to the solution by means of the graph?

8. Describe the solution of a system of three linear equations in three unknowns. Is it immaterial which of the three variables is eliminated first?

9. Can you find a definite solution for two equations each containing three unknowns? Illustrate this by means of the equations $4x-3y-z=5$ and $x+y+z=2$.

CHAPTER VI

SPECIAL PRODUCTS AND FACTORS

123. Repeated Factors. Number expressions containing repeated factors have already been considered in Chapter III. $x \cdot x$ was written x^2 and called *the square of* x, or x *square;* similarly, $(a+b)(a+b)$ was written $(a+b)^2$ and read *the square of the binomial* $a+b$ or *the binomial* $a+b$ *squared.*

124. Definitions. Any number written over and to the right of a number expression is called an **index** or **exponent** and, if a *positive integer,* shows how many times that expression is to be taken as a factor.

A product consisting entirely of equal factors is called a **power** of the repeated factor. The repeated factor is called the **base** of the power.

E.g. x^3 means $x \cdot x \cdot x$ and is read *the third power of* x or x *cube;* x^5 means $x \cdot x \cdot x \cdot x \cdot x$, and is read *the fifth power of* x or briefly x *fifth.* $(x-y)^3 = (x-y)(x-y)(x-y)$ and is read $(x-y)$ *cubed* or *the cube of the binomial* $(x-y)$.

Notice two important differences between an exponent and a coefficient.

(1) $5a = a+a+a+a+a$, while $a^5 = a \cdot a \cdot a \cdot a \cdot a$.

(2) In $5abc$ the coefficient 5 applies to the product abc, while in abc^5, the exponent 5 applies only to the factor c. In order to make it apply to the product it is necessary to use a parenthesis, thus, $(abc)^5$ means the product abc taken five times as a factor.

EXERCISES

Perform the following indicated multiplications:

1. $2^3, 2^4, 2^5, 2^6$.
2. $3^2, 3^3, 3^4, 3^5$.
3. $4^2, 4^3, 4^4, 4^5$.
4. $5^2, 5^3, 5^4, 5^5$.
5. $6^2, 6^3, 6^4$.
6. $7^2, 7^3, 7^4$.
7. $8^2, 8^3, 8^4$.
8. $9^2, 9^3, 9^4$.
9. $10^2, 10^3, 10^4, 10^5$.
10. $(a+b)^2$.
11. $(c-d)^2$.
12. $98^2=(100-2)^2$.
13. $(a+b+c)^2$.
14. $(a+b-c)^2$.
15. $(3-a)^2$.
16. $(3-b-c)^2$.
17. $(x-y-5)^2$.
18. $(w+z-4)^2$.
19. $(2-z-x+w)^2$.
20. $(x-y+5)^2$.
21. $(x-y)^3$.
22. $(x+y)^3$.
23. $(a+b)^4$.
24. $(c-d)^4$.

125. In the case of factors expressed in Arabic figures multiplications like the following may be carried out in either of two ways.

E.g. $3^2 \cdot 3^4 = 9 \cdot 81 = 729.$

or $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3^{2+4} = 3^6 = 729.$

But with literal factors the second process only is possible.

E.g. $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^{2+4} = a^6.$

The process in which the exponents are added applies only when the factors are powers of the same base.

E.g. $2^3 \cdot 3^2 = 8 \cdot 9 = 72$ cannot be found by adding the exponents.

2^1 or 2 is called the first power of 2.

Thus $2 \cdot 2^3 = 2^1 \cdot 2^3 = 2^{1+3} = 2^4.$

EXERCISES

In the following exercises carry out each indicated multiplication in two ways in case this is possible:

1. $2^5 \cdot 2^6$.
2. $2^2 \cdot 2^3$.
3. $2 \cdot 2^4$.
4. $3^2 \cdot 3^3$.
5. $3 \cdot 3^4$.
6. $3^2 \cdot 3^5$.
7. $4^2 \cdot 4^3$.
8. $4 \cdot 4^4$.

9. $5 \cdot 5^2$.
10. $5^2 \cdot 5^3$.
11. $6^2 \cdot 6^2$.
12. $7 \cdot 7^3$.
13. $a^2 \cdot a^3$.
14. $x^3 \cdot x^2$.
15. $x^7 \cdot x^4$.
16. $t^3 \cdot t^4$.
17. $t^2 \cdot t^3 \cdot t^4$.
18. $2^3 \cdot 2^2 \cdot 2^4$.
19. $3 \cdot 3^2 \cdot 3^3$.
20. $2^2 \cdot 2^3 \cdot 2^2 \cdot 2$.

126. Illustrative Problem. To multiply 2^k by 2^n, k and n being any two positive integers.

Solution. 2^k means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to k factors,
and 2^n means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to n factors.
Hence $2^k \cdot 2^n = (2 \cdot 2 \cdot 2 \cdots \text{to } k \text{ factors})\ (2 \cdot 2 \cdots \text{to } n \text{ factors})$
$= 2 \cdot 2 \cdot 2 \cdot 2 \cdots$ to $k + n$ factors in all.
That is, $$2^k \cdot 2^n = 2^{k+n}.$$

The preceding examples illustrate the following principle:

127. Principle XIV. *The product of two powers of the same base is found by adding the exponents of the powers and making this sum the exponent of the common base.*

EXERCISES

Perform the following indicated multiplications by means of Principle XIV:

1. $2^3 \cdot 2^7$.
2. $a^3 \cdot a^7$.
3. $3^4 \cdot 3^5$.
4. $x^4 \cdot x^5$.
5. $3^k \cdot 3^n$.
6. $x^k \cdot x^n$.
7. $4^a \cdot 4^b$
8. $3^{7a} \cdot 3^{2b}$.
9. $5^{2+n} \cdot 5^{2-n}$.
10. $7^{2x+y} \cdot 7^{2x-y}$.
11. $a^m \cdot a^n$.
12. $t^{2a} \cdot t^{3b+a}$.
13. $y^{4a} \cdot y^{3a}$.
14. $x^{3b} \cdot x^{2a+4b}$.
15. $w^x \cdot w^{y+3x}$.
16. $n^2 \cdot n^{3c+4b}$.
17. $c^x \cdot c^{2-x}$.
18. $x^{ac} \cdot x^{bc}$.
19. $r^{3c} \cdot r^{4c}$.
20. $s^{ax} \cdot s^{cx}$.
21. $v^{2x-1} \cdot v^{3x+2}$.

Perform the following multiplications by means of Principles IV, III, and XIV:

22. $2^3(2^2 + 2^4)$.
23. $2^2(3 \cdot 2^4 + 5 \cdot 2^3)$.
24. $4^4(3 \cdot 4^3 - 5 \cdot 4^2)$.
25. $a^2(a^3b - ab^3)$.
26. $x^{2a}(4\,x^a + 3\,x^{3a})$.
27. $r^s(5\,r^{2s} - 3\,r^s)$.

Multiply the following and state all principles used:

28. $a^2m^2 - b^2m^3$ by m^4.
29. $4 \cdot 3^2 - 5 \cdot 7 \cdot 2$ by 2^4.
30. $2 \cdot 3 - 4 \cdot 3^2 + 7 \cdot 3$ by 3.
31. $4x^2 - 3x^3 + 6x^4$ by x^2.
32. $5x - 3x^2 + 2x^4$ by x^4.
33. $3y^2 + 4y^4 - y^3$ by y.
34. $7x^4 - 5x^3 - 2x$ by x^3.
35. $3a^2b^4 + 2a^2b - 4a^2b^2$ by a^3.
36. $4a^{2b} - a^{3c} + a^{2d}$ by a^{2b}.
37. $3^2 \cdot 4^{2n} - 6^3 \cdot 4^{4n}$ by 4^{3n}.
38. $a^{2c}b^{3c} - a^cb^{2c}$ by a^{4c}.
39. $6^2 \cdot 4^3 - 7^4 \cdot 4^2 + 4^4$ by 4^3.
40. $12x^2y^3 - 6x^3y^2 + 3xy$ by x^3.
41. $2a^{3b} - 3a^{2c} - 4a^{4b}$ by a^{2c}.
42. $6x^{3a-2b} + 8x^{2b-3a}$ by x^{2a+2b}.
43. $y^{3m+2n} - y^{2n-3m+1}$ by y^{3m+3n}.
44. $x^{5c-3a} + x^{3c+2a} + x^{2a}$ by x^{4a-2c}.
45. $a^{nx-bx} - a^{bx-nx} - a^x$ by a^{bx+nx}.

PRODUCTS OF MONOMIALS

128. In multiplying two numbers each of which is in the factored form, if the factors are all expressed in Arabic figures, the operation may be carried out in two ways.

E.g. $(2 \cdot 3)(2 \cdot 3 \cdot 5) = 6 \cdot 30 = 180$.
Also $(2 \cdot 3)(2 \cdot 3 \cdot 5) = 2^2 \cdot 3^2 \cdot 5 = 4 \cdot 9 \cdot 5 = 180$.

In the second process one of the factors, 2 or 3, is multiplied in and this product is then multiplied by the other factor, 2 being combined with the factor 2 and 3 with 3 by Principles III and XIV. In the case of literal factors the second process only is available.

E.g. $(a^2b^2)(5a^4b^3c) = 5a^{4+2}b^{2+3}c = 5a^6b^5c$.

EXERCISES

Perform the following indicated multiplications, each in two ways where possible:

1. $(5 \cdot 7^2)(5^3 \cdot 7^3)$.
2. $(3^2 \cdot 5)(3 \cdot 2^3 \cdot 5^2)$.
3. $(4 \cdot 5^3)(5 \cdot 4^3)$.
4. $(3 \cdot 4^2)(4 \cdot 3^2)$.
5. $3ab(5a^2b^2)$.
6. $4x(3xy)$.
7. $5t(3st^2)$.
8. $7m(4m^2n)$.
9. $6k^2(6^2k^3)$.

By definition (§ 77), expressions in the factored form, such as $3ab$, $5a^2b^2$, $3 \cdot 2^3 \cdot 5^2$, etc., are monomials.

The preceding examples illustrate the following principle:

129. Principle XV. *The product of two monomials is found by multiplying either one by each factor of the other in succession.*

Each factor of the multiplier is associated with any desired factor of the multiplicand according to Principle III, and where the bases are the same the exponents are added according to Principle XIV.

If there are no factors common to multiplier and multiplicand, the product can only be indicated by writing all the factors of both in succession.

EXERCISES

Multiply:

1. $4 \cdot 7 \cdot 8 \cdot 9$ by $2 \cdot 3 \cdot 5$.
2. $3 \cdot 8 \cdot 2$ by $2 \cdot 5 \cdot 6$.
3. $2\,xyz$ by $3\,x^2yz$.
4. $6^3 \cdot 2^4 \cdot x^5$ by $6 \cdot 2^3 \cdot x$.
5. $3\,x^4y^2$ by $4\,xy^3$.
6. $5\,a^2b^3c$ by ab^4c.
7. $2\,x^3b^4c^2$ by $5\,xb^4c$.
8. $9\,x^ay^bz^c$ by $4\,x^{2a}y^{2b}z^c$.
9. $6\,x^{4n+1}y^{2n-4}z^n$ by $3\,x^{1-4n}y^{5-n}z^{6-n}$.
10. $3\,n^{2x+4}m^{y-5}r^{2x-1}$ by $n^{2-x}m^{y-1}r^x$.

In each of the following exercises state which of the Principles I–XV are used:

11. $3^2 \cdot 2^3(2^4 \cdot 3^2 - 4 \cdot 2^2 - 2 \cdot 3^2)$.
12. $6 \cdot 3^4(4^2 \cdot 2^2 \cdot 3^2 + 4^3 \cdot 2^4 - 5 \cdot 3^4)$.
13. $2\,x(4 + 7\,x^4 - 3\,x^2y)$.
14. $4\,yx(3\,y^2x^3 - y^3x^2 + y^4x^4)$.
15. $5\,a^2b^2(a^3 - b^3 + a^2b^2)$.
16. $4\,x^4y^3(3\,ax - 4\,by + 2\,xy)$.
17. $2\,x^a(x^a - xb - 3\,c)$.
18. $2\,yx^c(y - x^{2c} + 3\,y)$.
19. $4\,x^{2n+1}(x^{3n+1} - 4\,y^{2n+5} + 5\,x^{2n}y^4)$.
20. $4\,x^{2n+1}y^{n+2m}(x^ny^m - x^my^n + x^m + y^n)$.
21. $(a + b)^3(a - b)$.
22. $(a - b)^3(a + b)$.
23. $(a + b)^2(a - b)^2$.
24. $(a^3 + a^2b + ab^2 + b^3)(a - b)$.
25. $(a + b - c)(a + b + c)$.
26. $(3\,x - 2\,y - 1)(2\,x + y)$.

27. $(5a-3b)(6a^2+2b^2-1)$.
28. $(3a^2-2b^2-3)(4a+3b^3)$.
29. $(1+a+a^2)(1-a)$.
30. $(1-a+a^2-a^3)(1+a)$.
31. $(a+b)(a-b)(a^2+b^2)$.
32. $(a+b)(a^2-ab+b^2)$.
33. $(a-b)(a^2+ab+b^2)$.
34. $(x+y)(x-y)$.
35. $(100+1)(100-1)$.
36. $(100+2)(100-2)$.
37. $(x+3)(x+5)$.
38. $(x-3)(x-5)$.
39. $(t+8)(t-3)$.
40. $(x-2y)(x+3y)$.
41. $(x^2+x+1)(x^2-x+1)$.
42. $(x+y)(x^3-x^2y+xy^2-y^3)$.
43. $(x-y)(x^3+x^2y+xy^2+y^3)$.
44. $(x^2-xy+y^2)(x^2+xy+y^2)$.
45. $(x^2+2xy+y^2)(x-y)^2$.
46. $(x^2-y^2)(x^4+x^2y^2+y^4)$.
47. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
48. $(x^2+y^2)(x+y)(x-y)$.
49. $(x-y)(x^2+xy+y^2)(x^3+y^3)$.
50. $(a^n-b^n)(a^{2n}+a^nb^n+b^{2n})$.

FACTORS OF NUMBER EXPRESSIONS

130. The factors of numbers are of great importance in arithmetic. For instance, the multiplication table consists of pairs of factors whose products are committed to memory for constant use. Likewise in algebra the factors of certain special forms of number expressions are so important that they must be known at sight.

An expression containing no fractions is said to be prime if it has no integral factors except itself and 1.

Thus 2, 3, x, $x+2$, a^2+b^2, are prime expressions.

A prime expression may be factored by using fractions or radicals. See p. 214.

Thus $5 = 2 \cdot 2\frac{1}{2}$ and $2 = \sqrt{2} \cdot \sqrt{2}$.

Such factors are not included in what are here called prime factors.

131. Monomial Factors. If the terms of a polynomial contain a common factor, they may be combined with respect to this factor according to Principles I and II. The expression is thus changed into a product of a monomial and a polynomial.

Illustrative Examples.

1. $ax + ay = a(x + y)$, by Principle I.
2. $a^2 - ab = a(a - b)$, by Principle II.
3. $9 \cdot 8 + 3 \cdot 4 \cdot 5 = 3 \cdot 4(3 \cdot 2 + 5)$, by Principle I.
4. $6a^2b - 4ab^2 = 2ab(3a - 2b)$, by Principle II.
5. $5xy - 3x^2y + 4x^3y = xy(5 - 3x + 4x^2)$, by I and II.

Observe that factoring each term of a polynomial *does not factor the polynomial.*

E.g. $330 - 210 - 60$ is not factored by writing it $2 \cdot 3 \cdot 5 \cdot 11 - 2 \cdot 3 \cdot 5 \cdot 7 + 2^2 \cdot 3 \cdot 5$; but by adding the coefficients of the common factor $2 \cdot 3 \cdot 5$, thus,

$$2 \cdot 3 \cdot 5 \cdot 11 - 2 \cdot 3 \cdot 5 \cdot 7 + 2^2 \cdot 3 \cdot 5 = 2 \cdot 3 \cdot 5(11 - 7 + 2).$$

Likewise $10a^2bc - 15ab^2c + 20abc^2$ is not factored, although *each term* is in the factored form. But if $10a^2bc - 15ab^2c + 20abc^2$ is written in the form $5abc\ (2a - 3b + 4c)$, it is then factored.

These examples illustrate the factoring of a polynomial when it contains a monomial factor common to every term.

When such a common factor has been found the whole expression is then written in the form of a product by means of Principles I and II. The result may be checked by Principles IV and XV. If the terms of a polynomial which is to be factored contain a common factor, this should always be removed at the outset.

EXERCISES

Factor the following polynomials:

1. $8-12-18+48$.
2. $3\cdot 4\cdot 5-15-20+35$.
3. $3\cdot 11\cdot 4-22\cdot 2+44\cdot 6$.
4. $3^4\cdot 2^4-3\cdot 2^3-5\cdot 2^2$.
5. $6\cdot 5^4\cdot 7+3\cdot 5^3\cdot 2-5^4\cdot 9\cdot 2^2$.
6. $13\,a^4b-16\,a^3b^4-2\,a^3b^2$.
7. $15\,xy^4-20\,x^3y+x^2y^2$.
8. $9\,v^3w^4+21\,v^4w^3-18\,v^2w^2$.
9. $12\,a^4b^3-8\,a^3b^4-6\,a^2b^2$.
10. $32\cdot 3^4-64\cdot 3^3-16\cdot 2^3\cdot 3^2$.
11. $27\cdot 2^3+54\cdot 2^4-36\cdot 2^3$.
12. $11\,a^4x^2-44\,a^3x^4+33\,ax$.
13. $1\cdot 2\cdot 3\cdot 4+2\cdot 3\cdot 4\cdot 5-3\cdot 4\cdot 5\cdot 6+4\cdot 5\cdot 6$.
14. $72\,x^4b^3a-36\,x^4b^2a^2-48\,x^3b^3a^4$.
15. $84\,x^9b^7y^4+18\,x^3b^5y^4+12\,x^2b^3y^2$.
16. $19\cdot 3^4\cdot 2^4\cdot 5^3-13\cdot 3^4\cdot 2^3\cdot 5^2-29\cdot 3^2\cdot 2^2\cdot 5^4$.
17. $17\,a^4b^3c^2+51\,a^3b^4c^3-34\,a^2b^2c^4$.
18. $38\,a^{12}b^{14}c^4-76\,a^{11}b^{12}c^3-76\,a^{13}b^{11}c^2$.
19. $4\,x^{2a}y^{3b}+6\,x^{3a}y^{2b}-8\,x^{5a}y^{4b}$.
20. $3\,a^{4n+2}b^{3n+4}+6\,a^{6n+4}b^{3n+3}-12\,a^{5n+3}b^{5n+6}$.

TRINOMIAL SQUARES

132. In §§ 87 and 88 we found by multiplication:

$$(a+b)^2=a^2+2\,ab+b^2, \tag{1}$$

and

$$(a-b)^2=a^2-2\,ab+b^2. \tag{2}$$

By means of these formulas we may square the sum or difference of any two number expressions.

E.g. $(3\,x+2\,y)^2=(3\,x)^2+2\cdot 3\,x\cdot 2\,y+(2\,y)^2=9\,x^2+12\,xy+4\,y^2$.
Also $[(5+r)-(s-1)]^2=(5+r)^2-2(5+r)(s-1)+(s-1)^2$.

The last expression may now be reduced by performing the indicated operations.

In this manner write the squares of the following binomials. Verify the first ten results by actual multiplication.

1. $t+8$.
2. $r-12$.
3. $3x-5y$.
4. $6a-7$.
5. m^5+3n.
6. $7x^2+3x$.
7. $3y^3-2y^2$.
8. $8m^2n-7mn^2$.
9. $5a^3-3b^3$.
10. $4a^2x^2y^3-3a^3xy^4$.
11. $7(a-3)-2(b+c)$.
12. $4(x-y)+2(z+y^2)$.
13. $(3a-2b)+5$.
14. $7x-(4r-s)$.
15. $7(m^2-3)-6(m^3+n)$.
16. $3(z+y)-2(3+x)$.
17. $4(x^3-3y)+(x^2+4y^2)$.
18. $2x^2y-5(x-y)$.
19. $3(5x^2-z)-4x^2z^2$.
20. $7(r-s)+3(r^2s^2-2rs)$.

133. The binomial $a+b$ is one of the two equal factors of $a^2+2ab+b^2$, and is called the **square root** of this trinomial.

Likewise $a-b$ is the square root of $a^2-2ab+b^2$.

In each case a is the square root of a^2 and b of b^2. Hence $2ab$ is twice the product of the square roots of a^2 and b^2.

From the squares obtained in the last article, we learn to distinguish whether any given trinomial is a perfect square, as in the following examples:

1. x^2+4x+4 is in the form of (1), since x^2 and 4 are squares each with the sign $+$, and $4x$ is twice the product of the square roots of x^2 and 4. Hence

$$x^2+4x+4=x^2+2(2x)+2^2=(x+2)(x+2)=(x+2)^2.$$

2. x^2-4x+4 is in the form of (2), since it differs from (1) only in the sign of the middle term. Thus

$$x^2-4x+4=x^2-2(2x)+2^2=(x-2)(x-2)=(x-2)^2.$$

A trinomial which is the product of two equal factors is called a **trinomial square**.

EXERCISES

Determine whether the following are trinomial squares, and if so indicate the two equal factors. If any trinomial is not a square, make it so by modifying one of its terms.

1. $x^2 + 2xy + y^2$.
2. $x^2 - 2xy + y^2$.
3. $x^4 + 2x^2y^2 + y^4$.
4. $x^4 - 2x^2y^2 + y^4$.
5. $m^2 + n^2 - 2mn$.
6. $r^2 + s^2 + 2rs$.
7. $4x^2 - 8xy + 4y^2$.
8. $a^6 + b^6 + 2a^3b^3$.
9. $a^8 + b^8 - 2a^4b^4$.
10. $64 + t^2 - 16t$.
11. $16 + x^2 - 8x$.
12. $9 - 6y + y^2$.
13. $25x^2 + 16y^2 + 40xy$.
14. $4m^2 + n^2 + 2mn$.
15. $100 + s^2 + 20s$.
16. $64 + 49 + 112$.
17. $16a^2 + 25b^2 - 50ab$.
18. $16a^2 + 25b^2 + 40ab$.
19. $81 - 270b + 225b^2$.

134. From the foregoing examples we see that a trinomial is a perfect square if it contains two terms which are squares each with the sign $+$, while the third term, whose sign is either $+$ or $-$, is twice the product of the square roots of the other two. Then the square root of the trinomial is the sum or the difference of these square roots according as the sign of the third term is $+$ or $-$.

Since on multiplying we find $(a-b)^2$ and $(b-a)^2$ give the same result, we may write the factors of $a^2 - 2ab + b^2$ either $(a-b)(a-b)$ or $(b-a)(b-a)$. See page 214.

EXERCISES

Factor the following. If any one of the trinomials is found not to be a square, make it so by modifying one of its terms.

1. $9 + 2 \cdot 3 \cdot 4 + 16$.
2. $x^2 + 4y^2 + 4xy$.
3. $9x^2 + 18xy + 9y^2$.
4. $4x^2 + 4xy + y^2$.

5. $4x^2 + 8xy + 4y^2$.

6. $25x^2 + 12xy + 4y^2$.

7. $16x^2 + 16xy + 4y^2$.

8. $9r^2 + 36rs + 25s^2$.

9. $16x^6 + 8x^3y + y^2$.

10. $4x^8 + 12x^4a^3 + 9a^4$.

11. $a^{10} + 6a^5b + 9b^2$.

12. $(a + 1)^2 + 2(a + 1)b + b^2$.

13. $(x + 3)^2 + 4(x + 3)y + 4y^2$.

14. $x^6 + 12x^3 + 36$.

15. $a^4 + 18a^2 + 12$.

16. $121 + 4x^8 - 44x^4$.

17. $16x^4 + 64y^4 - 64x^2y^2$.

18. $81a^2 - 216a + 144$.

19. $4a^2 + 8ab^2 + 4b^2$.

20. $9b^4 + 18b^2c^4 + 9c^8$.

21. $4x^2 + 4y^2 - 8xy$.

22. $9a^2 - 16ab + 4b^2$.

23. $9x^4 - 24x^2b + 16b^2$.

24. $25 + 49x^2 - 70x$.

25. $-30ab^2 + 9a^2 + 25b^4$.

26. $16a^2 - 24ab + 9b^2$.

27. $36x^2 - 84x + 49$.

28. $25 - 90 + 81$.

29. $64x^2 - 32x + 9$.

30. $(3 + a)^2 + b^2 - 2b(3 + a)$.

31. $(2 - x)^2 - 2(2 - x)(x - 1) + (x - 1)^2$.

32. $(2 + y)^2 + 2(2 + y)(1 + 4) + (1 + 4)^2$.

33. $(3a - 2b)^2 - 10(3a - 2b) + 25$.

34. $(6a - b)^2 + (2a + 1)^2 - 2(6a - b)(2a + 1)$.

35. $25(a + b)^2 + 50(a + b)(a - b) + 25(a - b)^2$.

36. $x^2 + 12x(a + b + c) + 36(a + b + c)^2$.

37. $49(m - 3)^4 + 36(m + 1)^6 - 84(m - 3)^2(m + 1)^3$.

38. $16(x - y)^2 - 16(x - y)(x + y) + 4(x + y)^2$.

39. $-30(a + b)(a - b)^2 + 25(a - b)^4 + 9(a + b)^2$.

THE DIFFERENCE OF TWO SQUARES

135. By multiplication,

$$(a+b)(a-b)=a^2-b^2.$$

Translate this formula into words. By means of this formula the product of the sum and difference of any two number expressions may be found.

E.g. $(3a+2b)(3a-2b)=(3a)^2-(2b)^2=9a^2-4b^2.$

Also $(x+y-z)(x+y+z)=[(x+y)-z][(x+y)+z]$

$$=(x+y)^2-z^2.$$

The last expression may now be simplified by performing the operations indicated.

In this manner form the following products. Verify the first ten by actual multiplication.

1. $(4a+5b)(4a-5b)$.
2. $(24x+12y)(24x-12y)$.
3. $(16a^2b^3-3c)(16a^2b^3+3c)$.
4. $(5-6t^2)(5+6t^2)$.
5. $(3x-2y)(3x+2y)$.
6. $(x^3-y^3)(x^3+y^3)$.
7. $(3x^4-5y^4)(3x^4+5y^4)$.
8. $[x+(y-z)][x-(y-z)]$.
9. $(x^m+y^n)(x^m-y^n)$.
10. $(a^{2n+1}+b^{2n-1})(a^{2n+1}-b^{2n-1})$.
11. $[c-(a-b)][c+(a-b)]$.
12. $[x-(y+z)][x+(y+z)]$.
13. $(a+b+c)(a-b-c)$.
14. $(a-b+c)(a-b-c)$.
15. $(r-y-z)(r-y+z)$.
16. $(a+b+c)(a+b-c)$.
17. $[a+b+(c-d)][a+b-(c-d)]$.
18. $[x+y+(u+v)][x+y-(u+v)]$.
19. $[4x-(a-2b)][4x+(a-2b)]$.
20. $[a+2b-(x-y^2)][a+2b+(x-y^2)]$.
21. $(11b^3x-3bx^3)(11b^3x+3bx^3)$.

From the preceding examples we see that every binomial which is the difference between two perfect squares is composed of two binomial factors; namely, the sum and the difference of the square roots of these squares.

E.g. $16x^2 - 9y^2$ is the difference between two squares, $(4x)^2$ and $(3y)^2$. Hence we have

$$16x^2 - 9y^2 = (4x)^2 - (3y)^2 = (4x + 3y)(4x - 3y).$$

EXERCISES

Factor each of the following:

1. $x^2 - 4y^2$.
2. $9x^2 - 36y^2$.
3. $x^4 - b^2$.
4. $4x^2 - 9b^8$.
5. $16a^4 - 9b^4$.
6. $64 - b^2$.
7. $1 - b^2$.
8. $a^2 - 1$.
9. $1 - 9x^4$.
10. $4 - 36a^2$.
11. $1 - 64a^8$.
12. $144x^2b^4 - 1$.
13. $256a^4b^6 - c^2$.
14. $1 - (x + y)^2$.
15. $4 - (x - 2y)^2$.
16. $16a - 25ab^2$.
17. $49x - 4xy^2$.
18. $225 - 64x^2y^4$.
19. $576a - 144ay^2$.
20. $5^8 - 3^8$.
21. $x^4 - 81y^2$.

136. It is important to determine whether a given expression can be written as the difference of two squares.

E.g. $a^2 + b^2 + 2ab - c^2 = (a + b)^2 - c^2 = (a + b + c)(a + b - c)$.
Also, $c^2 - a^2 + 2ab - b^2 = c^2 - (a^2 - 2ab + b^2) = c^2 - (a - b)^2$
$= (c - a + b)(c + a - b)$.

EXERCISES

In the following determine whether each is the difference of two squares, and if so, factor it accordingly; if not, make it so by modifying one of its terms.

1. $x^2 - (y - z)^2$.
2. $(x - y)^2 - z^2$.
3. $a^2 + b^2 - 2ab - 4$.
4. $x^2 + y^2 - 2xy - z^2$.
5. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.
6. $a^2 - (b^2 + c^2 + 2bc)$.
7. $(2a - 5)^2 - (3a + 1)^2$.
8. $(3x^2 - y)^2 - (x + y)^2$.

9. $(3a-2b)^2-(8a+5b)^2$.

10. $(3m-4)^2-(2m+3)^2$.

11. $(2r+s)^2-(3r-s)^2$.

12. $81-(a+b+c)^2$.

13. $x^4+x^2+1-4a^2$.

14. $a^2-(x+2y)^2$.

15. $9x^2-(a-b)^2$.

16. $25m^2-(3r+2s)^2$.

17. $4c^2-(4a^2+12ab-9b^2)$.

18. $9a^2+6ab+b^2-c^2$.

19. $16x^2-a^2+4ab-4b^2$.

20. $a^2-2ab+b^2-c^2$.

21. $c^2-(a^2+2ab+b^2)$.

22. $c^2-(a^2-2ab+b^2)$.

23. $(a+b)^2-(a-b)^2$.

24. $a^2+4ab+b^2-x^2$.

25. $a^2+4ab+4b^2-x^2$.

26. $9a^2+16b^2-32ab-x^4$.

27. $a^2+4ab+4b^2-(x^2-2xy+y^4)$.

28. $(3x-2)^2-(4x^2+9y^2-12xy)$.

29. $x^2+4xy-y^2-(a^2+2ab+b^2)$.

30. $16x^4y^2-(4x^2+9y^2+12xy)$.

31. $(a+b)^2-(4a^2+9b^2-12bc)$.

THE SUM OF TWO CUBES

137. By multiplication we find

$$(a+b)(a^2-ab+b^2)=a^3+b^3.$$

The pupil should perform the multiplication indicated in this formula and carefully note how the terms cancel in the product.

By means of this formula obtain the following products:

1. $(x+y)(x^2-xy+y^2)$.

2. $(a+2)(a^2-2a+4)$.

3. $(3a+b)(9a^2-3ab+b^2)$.

4. $(1+4x)(1-4x+16x^2)$.

5. $(w^2+3a^2)(w^4-3a^2w^2+9a^4)$.

6. $(x^2+y^2)(x^4-x^2y^2+y^4)$.

7. $(5a+b)(25a^2-5ab+b^2)$.

8. $(1+5x^2)(1-5x^2+25x^4)$.

9. $(1+2xy)(1-2xy+4x^2y^2)$.

10. $(2x+3y)(4x^2-6xy+9y^2)$.

These products show that every binomial which is the sum of the cubes of two numbers is the product of two factors, one of which is the sum of the numbers, and the other is the sum of their squares minus their product.

E.g. (1) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(2) $8a^3 + 27b^3 = (2a)^3 + (3b)^3$

$$= (2a + 3b)(4a^2 - 2a \cdot 3b + 9b^2).$$

(3) $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

Notice the difference between the trinomial $x^2 - xy + y^2$ and the trinomial square $x^2 - 2xy + y^2$.

EXERCISES

Determine whether each of the following is the sum of two cubes, and if so, find the factors; if not, make it so by modifying one of its terms. Check the results by multiplication.

1. $x^3 + y^3$.
2. $a^3 + 8b^3$.
3. $27a^3 + b^3$.
4. $8a^3 + 1$.
5. $1 + 64x^3$.
6. $2^3 + 3^3$.
7. $125 + 729$.
8. $1 + 125x^6$.
9. $27x^6 + 1$.
10. $8a^3 + 27b^3$.
11. $8a^3 + 64b^3$.
12. $w^3x^6 + x^9a^3$.
13. $1 + 8a^3b^3$.
14. $64x^3 + 343$.
15. $1 + a^3$.
16. $a^3 + 9b^3$.
17. $125x^3 + y^6$.
18. $1 + x^8$.
19. $64x^2 + 27y^3$.
20. $8^3 + 10^3$.
21. $1 + 729x^6$.
22. $x^6 + y^{12}$.
23. $a^9 + b^3$.
24. $27r^3 + 125s^3$.

THE DIFFERENCE OF TWO CUBES

138. By multiplication, we find

$$(a - b)(a^2 + ab + b^2) = a^3 - b^3.$$

In performing this multiplication, note carefully how the terms cancel in the product.

By means of this formula obtain the following products.

1. $(x-y)(x^2+xy+y^2)$.
2. $(b-3)(b^2+3b+9)$.
3. $(2a-b)(4a^2+2ab+b^2)$.
4. $(1-4x)(1+4x+16x^2)$.
5. $(a^2-b^2)(b^4+a^2b^2+b^4)$.
6. $(w^2-2a^2)(w^4+2w^2a^2+4a^4)$.
7. $(3a-2b)(9a^2+6ab+4b^2)$.
8. $(3x^2-1)(9x^4+3x^2+1)$.
9. $(1-2xy)(1+2xy+4x^2y^2)$.
10. $(2x-3y)(4x^2+6xy+9y^2)$.

These products show that the difference of the cubes of two numbers is the product of two factors, one of which is the difference of the numbers, and the other the sum of the squares of the numbers plus their product.

E.g. (1) $x^3-y^3=(x-y)(x^2+xy+y^2)$.

(2) $8a^3-64b^3=(2a)^3-(4b)^3$
$=(2a-4b)(4a^2+2a\cdot 4b+16b^2)$.

(3) $a^9-b^9=(a^3)^3-(b^3)^3=(a^3-b^3)(a^6+a^3b^3+b^6)$.

Notice the difference between the factor x^2+xy+y^2, and the trinomial square $x^2+2xy+y^2$.

EXERCISES

Determine whether each of the following is the difference of two cubes, and if so, find the factors; if not, make it so by changing one of its terms. Check the results by multiplication.

1. a^3-b^3.
2. $8a^3-b^3$.
3. a^3-8b^3.
4. $8a^3-8b^3$.
5. 3^3-2^3.
6. $1-a^{10}$.
7. $1-8a^3$.
8. $64x^3-y^3$.
9. $27-125a^3$.
10. x^6-y^6.
11. $1-x^6$.
12. x^9-8.
13. $1-125x^3$.
14. $8-27x^3$.
15. $27x^3-64$.
16. 2^3x^3-1.
17. $8x^4-y^3$.
18. $64a^3-27b^3$.
19. $1-729x^6$.
20. x^6-y^{12}.
21. $27r^3-125s^3$.

22. Also factor 10, 11, 19, and 20 as the difference of two squares.

TRINOMIALS OF THE FORM $x^2+(a+b)x+ab$.

139. In §§ 82 to 85 were found such products as

$$(1)\ (x+5)(x+2)=x^2+7x+10.$$
$$(2)\ (x-5)(x-2)=x^2-7x+10.$$
$$(3)\ (x+5)(x-2)=x^2+3x-10.$$
$$(4)\ (x-5)(x+2)=x^2-3x-10.$$

In each case the binomials to be multiplied have one term x in common, and the other two terms unlike. The trinomial product in each case is the square of the term common to the binomials, the algebraic sum of the unlike terms times the common term, and the product of the unlike terms.

Thus, the coefficient of x in (1) is $5+2$, in (2) it is $(-5)+(-2)$, in (3) it is $(+5)+(-2)$, and in (4) it is $(-5)+(+2)$.

The third term of the product in (1) is $(+5)(+2)$, in (2) it is $(-5)(-2)$, in (3) it is $(+5)(-2)$, and in (4) it is $(-5)(+2)$.

With these points clearly in mind, such products may be written out at once, without the formal work of multiplication.

EXERCISES

Find the following products by inspection:

1. $(a+7)(a+9)$.
2. $(b-5)(b+7)$.
3. $(x-7)(x-17)$.
4. $(y+8)(y-3)$.
5. $(y+7)(y-9)$.
6. $(y-7)(y-1)$.
7. $(x-3)(x-4)$.
8. $(x-5)(x-9)$.
9. $(x-1)(x+11)$.
10. $(c-3)(c+12)$.
11. $(c^2-1)(c^2-5)$.
12. $(x^4-5)(x^4-3)$.
13. $(a^2-2)(a^2-4)$.
14. $(a^3-1)(a^3+4)$.
15. $(2a-1)(2a-3)$.
16. $(2a+3)(2a-4)$.

17. $(x-5)(x+2)$.
18. $(3a-7)(3a+2)$.
19. $(a^2-8)(a^2-4)$.
20. $(b^3-3)(b^3-4)$.
21. $(2x+a)(2x+b)$.
22. $(5c-4)(5c+5)$.
23. $(x^2y^2+9)(x^2y^2-7)$.
24. $(1-3a)(1-2a)$.
25. $(3bc+4)(3bc-7)$.
26. $(100+3)(100+5)$.
27. $(2a^n+b)(2a^n+c)$.
28. $(x+a)(x+b)$.
29. $(x+a)(x-b)$.
30. $(x-a)(x-b)$.
31. $(3x^k+2y)(3x^k-5y)$.

140. It is possible to recognize such products at sight, and thus to find the factors by inspection.

Illustrative Examples. Determine whether the following trinomials are of the kind just considered:

1. $x^2+7x+12$. The question is whether two numbers can be found such that their sum is $+7$ and their product 12. The numbers 3 and 4 answer these conditions. Hence,

$$x^2+7x+12=(x+3)(x+4).$$

2. $x^2-5x-14$. Since the product of the numbers sought is -14, one number must have the sign $-$ and the other $+$; and since their sum is -5, the one having the greater absolute value must have the sign $-$. Hence, the numbers are -7 and $+2$, and we have $x^2-5x-14=(x-7)(x+2)$.

3. $x^2-7x+12=(x-3)(x-4)$. Since $(-3)(-4)=+12$ and $(-3)+(-4)=-7$.

4. $x^2+4x-12=(x+6)(x-2)$. Since $(+6)(-2)=-12$ and $(+6)+(-2)=+4$.

It is to be noted that it is not always possible to find integers to fulfill these two conditions.

E.g. given x^2+5x+3. By inspection, it is easily seen that there are no two integers such that their sum is $+5$ and their product $+3$.

EXERCISES

Determine whether each of the following trinomials can be factored by inspection, and if so, find the factors; if not, modify one term so as to make such factoring possible.

1. $x^2+3x+2.$
2. $x^2+x-6.$
3. $x^2-x-6.$
4. $x^2-6x+8.$
5. $x^2+6x+8.$
6. $x^2-3x-8.$
7. $x^2+2x-8.$
8. $a^2-4a-32.$
9. $a^2+4a-32.$
10. $b^2+15b+56.$
11. $b^2+8b+15.$
12. $b^2-b-56.$
13. $b^2+b-56.$
14. $c^2-3c-15.$
15. $x^2-15x+56.$
16. $x^2+15x-54.$
17. $x^2-14x-95.$
18. $y^2+21y+98.$
19. $y^2-7y-98.$
20. $x^2-19x+78.$
21. $x^4+18x^2+77.$
22. $x^4-5a^2-104.$
23. $a^2+32a+240.$
24. $a^4-11a^2+28.$
25. $a^4-11a^2-60.$
26. $a^2-14a-51.$
27. $a^2-3a-54.$
28. $x^4-8x^2-32.$
29. $a^6-3a^3-154.$
30. $x^2-10x+25.$
31. $a^2b^6-13ab^3-30.$
32. $x^2-17xyz+72y^2z^2.$
33. $r^8+6r^4s-91s^2.$
34. $a^4c^4+9a^2c^2-162.$
35. $a^2+11a-210.$
36. $m^4+4m^2n^2+4n^2.$
37. $s^2t^2-15st-54.$
38. $a^2b^2-27ab+26.$
39. $l^2+13l+42.$
40. $x^2y^2-11xy-180.$
41. $9a^2+24a+16.$
42. $81a^2-99a+30.$
43. $g^2+26g+133.$
44. $x^2+5xy-84y^2.$
45. $r^2+3r-154.$
46. $u^2+38uv+165v^2.$
47. $(a+b)^2-19(a+b)+88.$
48. $(x-y)^2-14(x-y)+40.$
49. $(r-s)^2-17(r-s)+60.$
50. $x^2+(a+b)x+ab.$

TRINOMIALS OF THE FORM $ax^2 + bx + c$.

141. Find the product of $2x + 5$ and $3x + 2$.

$$\begin{array}{l} 2x+5 \\ \underline{3x+2} \\ 6x^2+15x \\ \underline{4x+10} \\ 6x^2+19x+10 \end{array}$$

The products $3x \cdot 2x$ and $2 \cdot 5$ are called *end products* and $2 \cdot 2x$ and $5 \cdot 3x$ are called *cross products*. In this case the cross products are similar with respect to x and are added. Hence the final product is a trinomial two of whose terms are the end products and the third term is the sum of the cross products.

This fact enables us to write such products at once.

E.g. $(3a+4)(5a-7) = 15a^2 - a - 28$.

In this case $15a^2$ is the first end product and -28 the second, while $-a$ is the sum of the two cross products, $20a$ and $-21a$.

In this manner obtain the following products:

1. $(2a+3)(a+3)$.
2. $(4a-1)(3a+2)$.
3. $(2x+5)(x-7)$.
4. $(7r+8)(3r-6)$.
5. $(2x+8)(9x-4)$.
6. $(3m-1)(4m+3)$.
7. $(5s-7)(2s-4)$.
8. $(2x-1)(7x+4)$.
9. $(4n-9)(5n-7)$.
10. $(8y-1)(5y+11)$.
11. $(3t-5)(t+4)$.
12. $(5x-y)(2x+3y)$.
13. $(3x-2y)(x+3y)$.
14. $(4a-3y)(a+y)$.
15. $(3r-2s)(2r+s)$.
16. $(5m-n)(2m+n)$.
17. $(5a+3x)(3a-4x)$.
18. $(4a-5b)(a+3b)$.
19. $(3a+5b)(a-b)$.
20. $(3c-7d)(2c+3d)$.

142. Trinomials in the form of the above products may sometimes be factored by inspection.

Ex. 1. Factor: $5x^2+16x+3$.

If this is the product of two binomials they must be such that the end products are $5x^2$ and 3 and the sum of the cross products $16x$.

One pair of binomials having the required end products is $5x+3$ and $x+1$, another is $5x-1$ and $x-3$, and still another $5x+1$ and $x+3$.

The sum of the cross products in the first pair is $8x$, in the second pair $-16x$, and in the third pair $16x$. Since the sum of the cross products in the last pair is the one required, the factors are $5x+1$ and $x+3$.

Ex. 2. Factor: $6a^2-5a-4$.

In this case as in the one preceding there are several pairs of binomials whose end products are $6a^2$ and -4, such as $2a-2$ and $3a+2$, $6a-1$ and $a+4$, etc. By trial we find that among these $3a-4$ and $2a+1$ is the only pair the sum of whose cross products is $-5a$. Hence $6a^2-5a-4=(3a-4)(2a+1)$.

Obviously not every trinomial of this form can be factored in this manner. Thus, for example, in $6a^2+5a+4$, no pair of binomials whose end products are $6a^2$ and 4 has the sum of its cross products $5a$.

In this manner factor the following:

1. $3x^2+5x+2$.
2. $9a^2+9a+2$.
3. $2x^2+11x+12$.
4. $9x^2+36x+32$.
5. $2x^2-x-28$.
6. $12s^2+11s+2$.
7. $6t^2+7t-3$.
8. $6x^2-x-2$.
9. $5r^2+18r-8$.
10. $14a^2-39a+10$.
11. $5x^2+26x-24$.
12. $2x^2-5x+2$.
13. $2m^2-m-3$.
14. $7c^2-3c-4$.
15. $5x^4+9x^2-18$.
16. $7a^4+123a^2-54$.

17. $6c^2-19c+15$.
18. $3a^2-21a+30$.
19. $6d^2+4d-2$.
20. $20a^2-a-99$.
21. $12c^2+25c+12$.
22. $8+6a-5a^2$.
23. $15-5x-10x^2$.
24. $6-5x-4x^2$.
25. $3h^2-13h+14$.
26. $15r^2-r-2$.
27. $2t^2+11t+5$.
28. $10-5x-15x^2$.
29. $5x^2-33x+18$.
30. $20-9x-20x^2$.

FACTORS FOUND BY GROUPING

143. Besides the methods of factoring which have been applied to the types of expressions thus far considered, there are various other processes which will be considered in the Advanced Course. One other method of general application will suffice here.

Ex. 1. Find the factors of $ax+ay+bx+by$.

By Principle I, the first two terms may be added and also the last two.

Thus, $$ax+ay+bx+by=a(x+y)+b(x+y).$$

The given expression has thus been changed into the sum of two *compound terms* which have a common factor, $(x+y)$, and may be added with respect to this factor by Principle I.

Thus, $$a(x+y)+b(x+y)=(a+b)(x+y).$$

Hence, $$ax+ay+bx+by=(a+b)(x+y).$$

Ex. 2. Factor $ax-ay-bx+by$.

Combining the first two terms with respect to a and the second two with respect to $-b$, we have,

$$ax-ay-bx+by=a(x-y)-b(x-y).$$

Again combining with respect to the factor $x-y$,

$$ax-ay-bx+by=(a-b)(x-y).$$

The success of this method depends upon the possibility of so grouping and combining the terms as to reveal a *common* binomial factor. The order of the terms in the given polynomial may be changed as found desirable.

EXERCISES

Factor the following expressions by means of Principles I and II:

1. $ab^2+ac^2-db^2-dc^2$.
2. $6\,ms-15\,nt+9\,ns-10\,mt$.
3. $8\,ax-10\,ay+4\,bx-5\,by$.
4. $2\,a^2+3\,ak-14\,an-21\,nk$.
5. $ac+bc+ad+bd$.
6. $ax^2-bx^2-ay^2+by^2$.
7. $8\,ac-20\,ad-6\,bc+15\,bd$.
8. $2\,ax-6\,bx+3\,by-ay$.
9. $5+4\,a-15\,c-12\,ac$.
10. $15\,b-6-20\,bc+8\,c$.
11. $2\,n^2-cn+2\,nd-cd$.
12. $5\,ax-15\,ay-3\,bx+9\,by$.
13. $3\,xa-12\,xc-a+4\,c$.
14. $3\,xy-4\,mn+6\,my-2\,xn$.
15. $7\,mn+7\,mr-2\,n^2-2\,nr$.
16. $a-1+a^3-a^2$.
17. $3\,s+2+6\,s^4+4\,s^3$.
18. $as^2-3\,bst-ast+3\,bt^2$.
19. $3\,mn+6\,m^2-2\,am-an$.
20. $2\,ar+2\,as+2\,br+2\,bs$.

MISCELLANEOUS EXERCISES

Classify the following expressions according to the foregoing types for factoring, and indicate the factors:

1. $x^2+5\,x+6$.
2. $1-x^3$.
3. $x^2+11\,x+30$.
4. $4\,x^2+9\,y^2+12\,xy$.
5. $4\,x^2+9\,y^2-12\,xy$.
6. $5\,x^2+4\,ax+7\,xy$.
7. $2\,n^2-6\,nc-3\,ny+9\,cy$.
8. $4\,x^2-y^2$.

9. a^3+b^3.
10. $9x^2+y^4+6xy^2$.
11. $2y^2a^3+4ya^2-8ya$.
12. x^2+7x+6.
13. $9x^2+36y^4+36xy^2$.
14. $9y-9z-2xy+2xz$.
15. a^3-1.
16. $a^4+b^4+2a^2b^2$.
17. a^4-25.
18. $27a^3-125$.
19. $4a^2+4ab+ab^2$.
20. $4a^2+9x^4-12ax^2$.
21. $1+x^3$.
22. $2x^2+5x+3$.
23. $36+4x^6+24x^3$.
24. $(x-1)^2-(x+1)^2$.
25. $8+64a^6$.
26. $ac-ax-4bc+4bx$.
27. $27-216a^3$.
28. $3^3+6^3a^3$.
29. $25(x+1)^2-4$.
30. $5cx-10c+4dx-8d$.
31. $4(x+2)^2+y^2+4(x+2)y$.
32. $ra+2rh-5sa-10sh$.
33. $-2a^2b+a^4+b^2$.
34. $2ha-hb+6a-3b$.
35. $3(a+1)^3+4(a+1)^2+a+1$.
36. $(x+a)^2-(x-a)^2$.
37. $15m^2+224m-15$.
38. $3x^2+27x+42$.
39. $x^4+49a^2+14ax^2$.
40. $27a^6-a^3x^3$.
41. $3^3a^6+a^3x^3$.
42. $8bd-40be+3cd-15ce$.
43. $x^2-11x+30$.
44. $(x+2)^2-4(x-2)^2$.
45. $x^4+9y^2-6yx^2$.
46. $4a^2-7ca^2-4d^2+7cd^2$.
47. $a^2+15a-16$.
48. $18-27c+16b-24cb$.
49. $4-(a^2+b^2-2ab)^2$.
50. $10r+3bs-6br-5s$.
51. $25+64x^6+80x^3$.
52. $1000-x^3$.
53. 10^3+x^3.
54. $8a^3+a^3b^3+b^2a^2$.
55. $100-49x^4$.
56. $100+625+500$.
57. $a^2-17a+72$.
58. $a^2+17a+72$.
59. a^2+16b^2-8ab.
60. x^6-y^9.

61. $24a^2+37a-72$.

62. x^4+15x^2-100.

63. 9^6+8^3.

64. $9x^4+16y^2+24x^2y$.

65. $1-1000$.

66. $16a^2b^2+24ab+36b^2$.

67. $64+8$.

68. $16a^2b^2+9a^2c^2+24a^2bc$.

69. $a^2+4b^2+4ab-4x^2$.

70. $a^3b^6+c^3$.

71. $5x^3+10x^3y^2+30x^3y^4$.

72. $16a^2c^2+4c^2x^2+16ac^2x$.

73. $a^6y^3-z^3$.

74. x^4-7x^2-120.

75. $9a^4b^2-12a^3b+4a^2$.

76. $8ab+27ab^7$.

77. $x^4+4x^2+4-x^6$.

78. $1-125a^3b^6$.

79. $16+16ab+4a^2b^2$.

80. $64a^3+8a^2b^3$.

81. $36a^2b^4+c^2b^4+12ab^4c$.

82. $4a^2+9b^4+12ab^2-16a^4$.

83. x^6+17x^3+30.

84. $25-(a^4-2a^2b^3+b^6)$.

85. $-112a^2c^3+49a^4+64c^6$.

86. $a^2-a-380$.

87. $24a^4c^4+a^8+144c^8a^2$.

88. $9x^2+4y^4-12xy^2-16$.

89. $81+100x^8-180x^4$.

90. a^4+27a^2+180.

91. a^4+3a^2-180.

92. a^4-3a^2-180.

93. $144-(a^4+b^2-2a^2b)$.

94. $81a^2b^4+49c^2-126ab^2c$.

95. $12s^2-23st+10t^2$.

96. $36x^4+12x^2y^4+y^8$.

97. $16x^6+9y^4+24x^3y^2-49$.

98. $y^2+35y+300$.

99. $5y^2-80y+300$.

100. $100-(16x^2+y^6-8xy^3)$.

101. $ac-bc+ad-bd$.

102. $6rd-15re+22cd-55ce$.

103. $z^3+ya-y^3z^3-ay^4$.

104. $x^4+2x^2+1-x^2$.

105. $60x^2+7xy-y^2$.

106. $x^2-20xy+75y^2$.

107. $x^2-17x-60$.

108. $65r^2+8r-1$.

109. $a^2-13a-140$.

110. $39x^4-16x^2+1$.

111. $625-(31-4a^2)^2$.

112. $36a^2-29ab+5b^2$.

113. $c^4 - 31c^2 + 220.$

114. $ac + d^5a - b^4c - b^4d^5.$

115. $26 + 39n - 22m - 33mn.$

116. $12x^2 + 11x - 56.$

117. $a^2 + 4ab + 4b^2 - (a^2 - 4ab + 4b^2).$

118. $(3x - 1)^2 - (x^2 + 4y^2 - 4xy)^2.$

119. $(x + 3y)^2 + (x - 2y)^2 + 2(x + 3y)(x - 2y).$

120. $16(a + b)^2 - 8(a - b)(a + b) + (a - b)^2.$

121. $256x^2 - (49x^2 + 4y^4 - 28xy^2).$

122. $(2x - a)^2 + 100(a - 3x)^2 + 20(2x - a)(a - 3x).$

123. $-48(a - b)(a + b) + 36(a - b)^2 + 16(a + b)^2.$

EQUATIONS SOLVED BY FACTORING

144. Illustrative Problem. There are two consecutive numbers the sum of whose squares is 61. What are the numbers?

Solution. Let $x =$ one of the numbers, then $x + 1$ is the other.

Hence,	$x^2 + (x + 1)^2 = 61$	(1)
By F,	$x^2 + x^2 + 2x + 1 = 61$	(2)
By F, I, S,	$2x^2 + 2x = 60$	(3)
By D,	$x^2 + x = 30$	(4)

These equations differ from any which we have studied heretofore in that they contain the squares of the unknown number, which cannot be removed by addition or subtraction.

It is evident on inspection that $x = 5$ satisfies equation (4).

That is, $5^2 + 5 = 30.$

Hence 5 is one of the numbers sought, and $5 + 1$ is the other, and these numbers satisfy the conditions of the problem,

Since $5^2 + 6^2 = 25 + 36 = 61.$

145. Definition. Equations which involve the second but no higher degree of the unknown number are called **quadratic equations.**

One method of solving quadratic equations is now to be considered.

By S, equation (4) above may be written

$$x^2 + x - 30 = 0. \tag{5}$$

Factoring the left member,

$$(x + 6)(x - 5) = 0. \tag{6}$$

Equation (6) is satisfied by any value of x which, substituted in the left member, reduces it to zero; and, since the product of two factors is zero if either factor is zero, we seek values of x which make $x - 5 = 0$ and also $x + 6 = 0$. Hence the equation is satisfied by $x = 5$ and also by $x = -6$. Thus,

$$(5 + 6)(5 - 5) = 11 \cdot 0 = 0,$$

and also $$(-6 + 6)(-6 - 5) = 0 \cdot -11 = 0.$$

Therefore -6 and $-6 + 1 = -5$ are two numbers which meet the condition of the problem.

That is, $$(-6)^2 + (-5)^2 = 36 + 25 = 61.$$

Hence this problem has two solutions, namely, the numbers 5 and 6 and the numbers -6 and -5.

Illustrative Problem. A rectangular flower bed 10 feet long and 6 feet wide is surrounded by a gravel walk whose area is 192 square feet. How wide is the walk?

Solution. Let the width of the walk be x feet, then

$2 \cdot 10x$ is the area of the sides,
$2 \cdot 6x$ is the area of the ends,
$4x^2$ is the area of the corners.

Hence the total area is

$$4x^2 + 2 \cdot 10x + 2 \cdot 6x = 192. \tag{1}$$

By F, $$4x^2 + 32x = 192. \tag{2}$$

By D, $$x^2 + 8x = 48. \tag{3}$$

By S, $$x^2 + 8x - 48 = 0. \quad (4)$$

Factoring, $$(x + 12)(x - 4) = 0. \quad (5)$$

But (5) is satisfied if $x + 12 = 0$, and also if $x - 4 = 0$. (6)

Hence $$x = -12 \text{ and also } x = 4. \quad (7)$$

Check by substituting in equation (3):

$$(-12)^2 + 8(-12) = 144 - 96 = 48.$$

Also $$4^2 + 8 \cdot 4 = 16 + 32 = 48.$$

Hence the quadratic equation to which this problem gives rise has two solutions, but since the width of a walk cannot be a negative number, only the number 4 satisfies the conditions of the problem.

Ex. 1. Solve the quadratic equation

$$5x^2 + 30x + 3 = 3 - 5x. \quad (1)$$

By A, S, $$5x^2 + 35x = 3 - 3 = 0. \quad (2)$$

Factoring, $$5x(x + 7) = 0. \quad (3)$$

But (3) is satisfied if $5x = 0$, and also if $x + 7 = 0$. (4)

Hence $$x = 0, \text{ and also } x = -7. \quad (5)$$

Check. Substitute $x = 0$ and also $x = -7$ in (1).

Ex. 2. Solve the quadratic equation

$$6x^2 + 11x = 10. \quad (1)$$

By S, $$6x^2 + 11x - 10 = 0. \quad (2)$$

Factoring, $$(3x - 2)(2x + 5) = 0. \quad (3)$$

But (3) is satisfied if $3x - 2 = 0$ and also if $2x + 5 = 0$.

Hence $$x = \tfrac{2}{3} \text{ and } x = -\tfrac{5}{2}.$$

Check by substituting each of these values in (1).

Ex. 3. Solve the quadratic equation

$$3x^2 - 5x - 7 = 2x^2 - x - 11. \qquad (1)$$

By A and S, $$x^2 - 4x + 4 = 0. \qquad (2)$$

Factoring, $$(x-2)(x-2) = 0. \qquad (3)$$

It follows from the first factor and also from the second that (3) is satisfied if $x = 2$.

In this case the two solutions turn out to be identical, while in Example 1 one solution was zero and the other was -7.

If we count each of these results as two solutions, then for every quadratic equation thus far solved we have found two values of the unknown number.

146. The method of solution above explained consists of three steps:

(1) Transform the equation so that all terms are collected in one member, with similar terms united, leaving the other member zero. This can always be done by Principle VIII. It is convenient to make the right member zero.

(2) Factor the expression on the left.

(3) Find the value of x which makes each of these factors zero. This is readily done by setting each factor equal to zero and solving it for the unknown.

EXERCISES

Find two solutions for each of the following quadratic equations:

1. $x^2 - 3x + 2 = 0$.
2. $x^2 + 7x = 30$.
3. $a^2 - 11a = -30$.
4. $a^2 + 13a = 30$.
5. $a^2 + 10a + 8 = -3a - 34$.
6. $a^2 + 3a = 10a + 18$.
7. $a^2 + 10a = -24 - 4a$.
8. $2x^2 - 6x = -40 + 12x$.
9. $3x + x^2 = 20x - 72$.
10. $17x + 30 = -x^2 - 40$.

11. $7x^2+2x=30x-21.$

12. $11x+3x^2=20.$

13. $16-5x+x^2=-2x^2-20x-2.$

14. $x^2-16=0.$

15. $x^2-1=0.$

16. $x^2-x=0.$

17. $x^2+x=0.$

18. $4x^2=25.$

19. $x^2+4x+4=0.$

20. $x^2+8x+16=0.$

21. $x^2+12x+6=5x-4.$

22. $2x^2-7x=60+7x.$

23. $60x+4x^2+144=8x.$

24. $18x=63-x^2.$

25. $24x^2=12x+12.$

26. $2x=63-x^2.$

27. $22x+x^2=363.$

28. $3x^2+7x=6.$

29. $2x^2=2-3x.$

30. $x-2=-3x^2.$

147. It is sometimes possible to solve other equations than quadratics by the above process.

Ex. 1. Solve the equation:

$$x^3+30x=11x^2. \quad (1)$$

By S, $$x^3-11x^2+30x=0. \quad (2)$$

By § 131, $$x(x^2-11x+30)=0. \quad (3)$$

By § 140, $$x(x-5)(x-6)=0. \quad (4)$$

(4) is satisfied if $x=0$, if $x-5=0$, and if $x-6=0$.
Hence the solutions are $x=0$, $x=5$, $x=6$.

Ex. 2. $x(x+1)(x-2)(x+3)=0.$

Any value of x which makes one of these factors zero reduces the product to zero and hence satisfies the equation. Hence the solutions of the equation are found from

$$x=0,\ x+1=0,\ x-2=0, \text{ and } x+3=0.$$

That is $x=0$, $x=-1$, $x=2$, $x=-3$ are the values of x which satisfy the equation.

Notice that this process is applicable only when one member of the equation is *zero* and the other member is *factored.*

EXERCISES

Solve the following equations by factoring:

1. $x^3 - x^2 = 6x.$
2. $5x = 4x^2 + x^3.$
3. $x^3 - 25x = 0.$
4. $x^3 - 3x^2 = -2x.$
5. $3x^3 = 15x^2 + 42x.$
6. $5x^3 + 315x = 80x^2.$
7. $x^2 + ax + bx + ab = 0.$
8. $x^2 - ax - bx + ab = 0.$
9. $4(x-2)^2 - (x+3)^2 = 0.$
10. $x^2 - ax + bx - ab = 0.$
11. $x^2 + ax - bx - ab = 0.$
12. $9(x+2)^2 - 4(x-3)^2 = 0.$
13. $x^3 - x - 3x^2 + 3 = 0.$
14. $x^3 - 4x - 8x^2 + 32 = 0.$
15. $5x^3 + 120x^2 = 119x - 2x^3 + 8x^2.$
16. $(x^2 + 6x - 16)(x^2 - 7x + 12) = 0.$
17. $(x+7)(x^2 - 17x + 72) = 0.$
18. $(x - 15)(x^3 + 11x^2 + 30x) = 0.$

148. Illustrative Problem. The paving of a square court costs 40 ¢ per square yard and the fence around it costs \$ 1.50 per linear yard. If the total cost of the pavement and the fence is \$ 100, what is the size of the court?

Solution. Let x = the length of one side in yards.

Then $40x^2$ = cost in cents of paving the court,

$150 \cdot 4x = 600x$ = cost of the fence in cents.

$$40x^2 + 600x = 10000. \quad (1)$$

By *D*,
$$x^2 + 15x = 250. \quad (2)$$

By *S*,
$$x^2 + 15x - 250 = 0. \quad (3)$$

Factoring,
$$(x-10)(x+25) = 0. \quad (4)$$

Whence,
$$x = 10, \text{ and also } x = -25. \quad (5)$$

It is clear that the length of a side of the court cannot be -25 yards. Hence 10 is the only one of these two results which has a meaning in this problem.

It happens frequently when a quadratic equation is used to solve a problem that one of the two numbers which satisfy this equation will not satisfy the conditions of the problem.

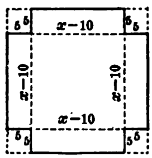
PROBLEMS

In each of the following problems find all the solutions possible for the equations and then determine whether or not each solution has a reasonable interpretation in the problem.

1. The dimensions of a picture inside the frame are 12 by 16 inches. What is the width of the frame if its area is 288 square inches?

2. There are two consecutive integers such that the sum of their squares is 3961. What are the numbers?

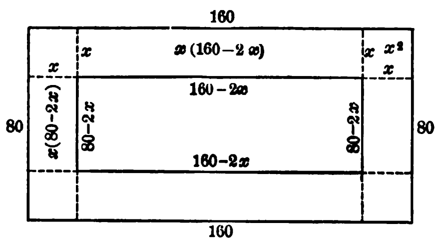
3. An open box is made from a square piece of tin by cutting out a 5-inch square from each corner and turning up the sides. How large is the original square if the box contains 180 cubic inches?



If x = length of a side of the tin, then the volume of the box is: $5(x-10)(x-10)=180$. (See the figure.)

4. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

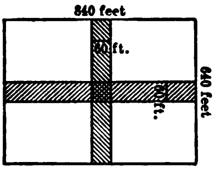
5. A farmer has a rectangular wheat field 160 rods long by 80 rods wide. In cutting the grain, he cuts a strip of equal width around the field. How many acres has he cut when the width of the strip is 8 rods?



6. How wide is the strip around the field of problem 5, if it contains $27\frac{1}{2}$ acres?

7. In the northwest a farmer using a steam plow starts plowing around a rectangular field 640 by 320 rods. If the strip plowed the first day lacks 16 square rods of being 24 acres, how wide is it?

8. A rectangular piece of ground 840 by 640 feet is divided into 4 city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?



9. A farmer lays out two roads through the middle of his farm, one running lengthwise of the farm and the other crosswise. How wide are the roads if the farm is 320 by 240 rods, and the area occupied by the roads is 1671 square rods?

QUADRATIC AND LINEAR EQUATIONS

149. When two simultaneous equations are given, one quadratic and one linear, they may be solved by the process of substitution, which was used (§ 116) in the case of two linear equations.

Illustrative Example. Solve the equations:

$$\begin{cases} x^2 - y^2 = -16. & (1) \\ x - 3y = -12. & (2) \end{cases}$$

From (2) by S, $\quad x = 3y - 12. \quad (3)$

Substituting (3) in (1), $\quad (3y - 12)^2 - y^2 = -16. \quad (4)$

From (4) by F, $\quad 9y^2 - 72y + 144 - y^2 = -16. \quad (5)$

From (5) by F, A, $\quad 8y^2 - 72y + 160 = 0. \quad (6)$

By D, $\quad y^2 - 9y + 20 = 0. \quad (7)$

Factoring, $\quad (y - 5)(y - 4) = 0. \quad (8)$

Hence, $\quad y = 5$, and $y = 4. \quad (9)$

Substitute $y = 5$ in (2) and find $x = 3$.

Substitute $y = 4$ in (2) and find $x = 0$.

Therefore (1) and (2) are satisfied by the two pairs of values,

$$x = 3,\ y = 5, \text{ and } x = 0,\ y = 4.$$

Check by substituting these pairs of values in (1) and (2).

EXERCISES

In the manner just illustrated solve the following:

1. $\begin{cases} x + 2y = 8, \\ 5x^2 + 12y^2 = 128. \end{cases}$

2. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 1. \end{cases}$

3. $\begin{cases} 2x - y = 6, \\ 4x^2 + 5y^2 = 36. \end{cases}$

4. $\begin{cases} x + 3y = 6, \\ x^2 + 3y^2 = 12. \end{cases}$

5. $\begin{cases} x - 2y = -2, \\ x^2 - 6y^2 = 10. \end{cases}$

6. $\begin{cases} 8x - 16y = -120, \\ 7x^2 + 2y^2 = 585. \end{cases}$

7. $\begin{cases} 7x + 9y = 88, \\ 7x^2 + 9y^2 = 736. \end{cases}$

8. $\begin{cases} x - y = 6, \\ x^2 - 7y^2 = 36. \end{cases}$

9. $\begin{cases} 3x + 2y = 7, \\ 3x^2 + 8y^2 = 35. \end{cases}$

10. $\begin{cases} x - 8y = -11, \\ 3x^2 - 16y^2 = 11. \end{cases}$

11. $\begin{cases} x - y = -7, \\ 4x^2 + 3y^2 = 147. \end{cases}$

12. $\begin{cases} x - y = 2, \\ x^2 - 5y^2 = 4. \end{cases}$

13. $\begin{cases} x - y = 1, \\ 3x^2 - 2y^2 = -5. \end{cases}$

14. $\begin{cases} 5x - 7y = -28, \\ 15x^2 + 49y^2 = 784. \end{cases}$

15. $\begin{cases} 6x - 7y = 18, \\ 36x^2 - 7y^2 = 324. \end{cases}$

16. $\begin{cases} x - 9y = 2, \\ x^2 - 45y^2 = 4. \end{cases}$

17. $\begin{cases} x + y = 8, \\ 13x^2 + 3y^2 = 160. \end{cases}$

18. $\begin{cases} 2x - 5y = -16, \\ 4x^2 + 15y^2 = 256. \end{cases}$

19. $\begin{cases} 7x + 4y = 7, \\ 49x^2 - 8y^2 = 49. \end{cases}$

20. $\begin{cases} x - 3y = -12, \\ x^2 - y^2 = -16. \end{cases}$

150. Illustrative Problem. The fence around a rectangular field is 280 rods long. What are the dimensions of the field, if its area is 30 acres?

Solution. Let $w =$ the width of the field in rods,

and $l =$ length of field in rods.

Then
$$2l + 2w = 280, \qquad (1)$$
and since one acre = 160 square rods,
$$lw = 4800. \qquad (2)$$

From (1), $l + w = 140$, and $w = 140 - l$.

Substituting this value of w in (2),

$$l(140 - l) = 4800,$$

or, $$l^2 - 140l + 4800 = 0.$$

Then, $$(l - 60)(l - 80) = 0.$$

Whence 60 and 80 both satisfy the equation.

If $l = 60$, then from equation (1) $w = 80$, and if $l = 80$, then $w = 60$.

We group these pairs of numbers as follows:

$$\begin{cases} l = 60, \\ w = 80, \end{cases} \text{and} \begin{cases} l = 80, \\ w = 60. \end{cases}$$

Substituting these pairs of values in both (1) and (2), we have

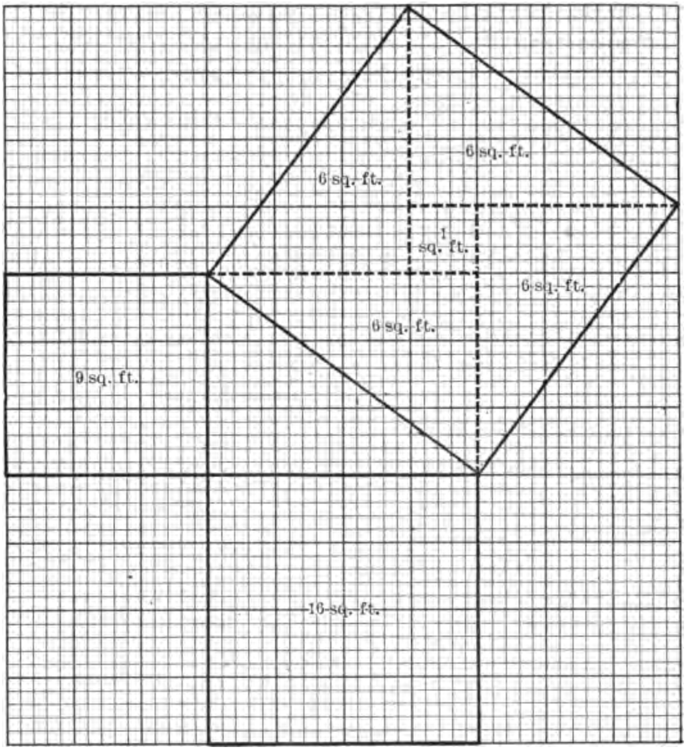
$$\begin{cases} 2 \cdot 60 + 2 \cdot 80 = 280, \\ 60 \cdot 80 = 4800, \end{cases} \text{and} \begin{cases} 2 \cdot 80 + 2 \cdot 60 = 280, \\ 80 \cdot 60 = 4800. \end{cases}$$

Hence, we obtain two pairs of numbers which satisfy both of these equations. The solution $l = 60$ and $w = 80$ is applicable to this problem only by calling the greater side the *width* instead of the length. We have seen before that solving a quadratic often results in one solution which is without meaning in the problem that gives rise to the equation.

In any case each solution should be carefully examined to ascertain whether under any reasonable interpretation they are both applicable to the problem.

151. If squares are constructed on the two sides, and also on the hypotenuse of a right-angled triangle, then the sum of the squares on the sides is equal to the square on the hypotenuse. This is proved in geometry, but may be verified by counting squares in the accompanying figure. This proposition was first discovered by the great philosopher and mathematician Pythagoras, who lived about 550 B.C. Hence it is called the Pythagorean proposition.

We now proceed to solve some problems by this proposition.



PROBLEMS

1. The sum of the sides about the right angle of a right triangle is 35 inches, and the hypotenuse is 25 inches. Find the sides of the triangle.

Solution. Let $a =$ the length of one side in inches,

and $b =$ the length of the other.

Then $a + b = 35,$ (1)

and $a^2 + b^2 = 25^2 = 625$ (Pythagorean proposition). (2)

From (1), $a = 35 - b.$

Substituting in (2), $(35 - b)^2 + b^2 = 625,$

or $1225 - 70\,b + b^2 + b^2 = 625,$

$$2\,b^2 - 70\,b + 600 = 0,$$

$$b^2 - 35\,b + 300 = 0,$$

$$(b - 20)(b - 15) = 0.$$

Whence $b = 20$, and $b = 15$.

From (1), if $b = 20$, $a = 15$, and if $b = 15$, $a = 20$; that is, the sides of the triangle are 15 and 20.

2. The difference between the two sides of a right triangle is 2 feet, and the length of the hypotenuse is 10 feet. Find the two sides.

3. The sum of the length and width of a rectangle is 17 rods, and the diagonal is 13 rods. Find the dimensions of the rectangle.

4. A room is 3 feet longer than it is wide, and the length of the diagonal is 15 feet. Find the dimensions of the room.

5. The length of the molding around a rectangular room is 46 feet, and the diagonal of the room is 17 feet. Find its dimensions.

6. The longest rod that can be placed flat on the bottom of a certain trunk is 45 inches. The trunk is 9 inches longer than it is wide. What are the dimensions of the bottom?

7. The floor space of a rectangular room is 180 square feet, and the length of the molding around the room is 56 feet. What are the dimensions of the room?

8. A rectangular field is 20 rods longer than it is wide, and its area is 2400 square rods. What are its dimensions?

9. A ceiling requires 24 square yards of paper, and the border is 20 yards long. What are the dimensions of the ceiling?

10. The area of a certain triangle is 18 square inches, and the sum of the base and altitude is 12. Find the base and altitude.

11. The altitude of a certain triangle is 7 inches less than the base, and the area is 130 inches. Find the base and altitude.

12. The sum of two numbers is 17, and the sum of their squares is 145. Find the numbers.

13. The difference of two numbers is 8, and the sum of their squares is 274. Find the numbers.

14. The difference of two numbers is 13, and the difference of their squares is 481. Find the numbers.

15. The sum of two numbers is 40, and the difference of their squares is 320. Find the numbers.

16. The sum of two numbers is 45, and their product is 450. Find the numbers.

17. The difference of two numbers is 32, and their product is 833. What are the numbers?

CHAPTER VII

QUOTIENTS AND SQUARE ROOTS

QUOTIENT OF TWO POWERS OF THE SAME BASE

152. Illustrative Problem. To divide x^6 by x^4.

Since by § 66 the quotient times the divisor equals the dividend, we seek an expression which multiplied by x^4 equals x^6.

Since by Principle XIV two powers of the same base are multiplied by adding their exponents, the expression sought must be that power of x whose exponent added to 4 equals 6. Hence the exponent of the quotient is $6 - 4 = 2$. That is, $x^6 \div x^4 = x^{6-4} = x^2$.

EXERCISES

Perform the following indicated divisions:

1. $2^4 \div 2^2$.
2. $2^3 \div 2^2$.
3. $2^4 \div 2$.
4. $3^3 \div 3^2$.
5. $3^4 \div 3$.
6. $3^4 \div 3^2$.
7. $9^{11} \div 9^2$.
8. $5^{13} \div 5^{12}$.
9. $7^{24} \div 7^{22}$.
10. $8^3 \div 8$.
11. $6^4 \div 6^2$.
12. $a^3 \div a^2$.
13. $a^4 \div a^3$.
14. $m^4 \div m^2$.
15. $x^4 \div x^2$.
16. $t^{14} \div t^4$.
17. $m^3 \div m$.
18. $n^6 \div n^2$.
19. $(20)^4 \div (20)$.
20. $(101)^{14} \div (101)^{13}$.
21. $41^7 \div 41^6$.

153. The process of division by subtracting exponents leads in certain cases to strange results.

Thus, according to this process, $x^4 \div x^4 = x^{4-4} = x^0$, which is as yet without meaning, since an exponent has been defined only when it is a *positive integer*. It cannot indicate, as in the case of a positive integral exponent, how many times the base is used as a factor. We know, however, that $x^4 \div x^4 = 1$, since any number divided by itself

equals unity. Hence if we use the symbol x^0 it must be interpreted to mean 1, no matter what number x represents. It is sometimes convenient in algebraic work to use it in this way.

Again by this process $x^2 \div x^4 = x^{2-4} = x^{-2}$, which is as yet without meaning, since negative exponents have not been defined. Cases of this kind are considered in the Advanced Course.

The preceding exercises illustrate the following principle:

154. Principle XVI. *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

For the present only those cases are considered in which the exponent of the dividend is greater than or equal to that of the divisor.

Notice that Principle XVI does not apply to powers of different bases.

E.g. $3^7 \div 2^4$ does not equal any integral base to the power, $7 - 4$. This division can be performed only by first multiplying out both dividend and divisor.

EXERCISES

Perform the following indicated divisions by means of Principle XVI:

1. $2^7 \div 2^3$.
2. $a^7 \div a^3$.
3. $3^4 \div 3^2$.
4. $x^4 \div x^2$.
5. $3^{3n} \div 3^{2n}$.
6. $x^{4n} \div x^{2n}$.
7. $3^{2a-1} \div 3^{a-2}$.
8. $5^{n+5} \div 5^{n+2}$.
9. $x^{a+4} \div x^{a+2}$.
10. $t^{4a} \div t^a$.
11. $x^{2a+b} \div x^{a+b}$.
12. $w^{2x} \div w^x$.
13. $(17)^{14} \div (17)^{13}$.
14. $4^3 \div 4$.
15. $(12)^4 \div (12)^3$.

In the following use Principles V, VI, and XVI:

16. $(2^4 + 2^3) \div 2^3$.
17. $(3 \cdot 2^4 + 5 \cdot 2^3) \div 2^2$.
18. $(3 \cdot 4^3 - 5 \cdot 4^4) \div 4^2$.
19. $(a^3b - a^4b^2) \div a^2$.
20. $(4x^3 + 3x^4) \div x^2$.
21. $(a^2m^4 - b^2m^3) \div m^3$.
22. $(4 \cdot 3^2 - 3^3 \cdot 5 \cdot 7) \div 3^2$.
23. $(2^3 \cdot 3 + 2^4 \cdot 3^3 - 2^3) \div 2^3$.
24. $(12x^3y - 11x^2y^2 + 5x^4) \div x^2$.
25. $(x^{3m+4} + x^{2m+3} - 5x^{m+2}) \div x^{m+1}$.

DIVISION OF MONOMIALS

155. In finding the quotient of two numbers each in the factored form, if the factors are represented by Arabic figures, the operation may be carried out in either of two ways.

E.g. $2^3 \cdot 3^3 \cdot 5 \div 2^2 \cdot 3 = 1080 \div 12 = 90.$

Also $2^3 \cdot 3^3 \cdot 5 \div 2^2 \cdot 3 = 2 \cdot 3^2 \cdot 5 = 90.$

In the second process we divide by one of the factors, 2^2 or 3, and divide this result by the other. $2^3 \cdot 3^3 \cdot 5$ divided by 2^2 gives, according to Principle V, $2 \cdot 3^3 \cdot 5$, and this result divided by 3 gives by the same principle $2 \cdot 3^2 \cdot 5$. In practice such operations may readily be performed simultaneously.

In the case of literal factors the second process only is available.

E.g. $5\,a^4b^3c \div a^2b^2 = 5\,a^{4-2}b^{3-2}c = 5\,a^2b^1c = 5\,a^2bc.$

EXERCISES

Perform the following indicated divisions in two ways when possible:

1. $5^3 \cdot 7^9 \div 5 \cdot 7^2.$
2. $2^3 \cdot 3^3 \cdot 5^3 \div 2 \cdot 3^2 \cdot 5.$
3. $4^4 \cdot 5^4 \div 4 \cdot 5^3.$
4. $3^4 \cdot 5^2 \div 3 \cdot 5.$
5. $15\,a^3b^4 \div 5\,ab.$
6. $12\,x^2y \div 4\,x.$
7. $18\,st^3 \div 3\,t^2.$
8. $28\,m^3n \div 7m.$

The preceding examples illustrate the following principle:

156. Principle XVII. *The quotient of two monomials is found by dividing the dividend by each factor of the divisor in succession.*

Each factor of the divisor is associated with any desired factor of the dividend according to Principle V, and when the bases are the same the exponents are subtracted according to Principle XVI.

157. If there are factors in the divisor not found in the dividend, this process terminates before the operation is completed. The remaining steps of the division must then be indicated, which is usually done in the form of a fraction.

E.g. $$x^2 \div 3x^3y = \frac{x^2}{3x^2 \cdot xy} = \frac{1}{3xy}.$$

Again, $$15a^3b^2c \div 3a^2bx^2y = \frac{15a^3b^2c}{3a^2bx^2y} = \frac{5abc}{x^2y}.$$

By this process all factors common to dividend and divisor have been canceled. Notice that in the first example the factor 1 remains when x^2 of the dividend is divided by x^2 of the divisor.

EXERCISES

Divide:

1. $4 \cdot 7 \cdot 9$ by $2 \cdot 3$.
2. $12 \cdot 8 \cdot 20$ by $2 \cdot 4 \cdot 5$.
3. $6x^3y^2z$ by $2xyz$.
4. $6^4 \cdot 3^4 \cdot x^3$ by $6^2 \cdot 5^2 \cdot x^2$.
5. $12x^{12}y^{13}$ by $4xy^3z$.
6. $5a^4b^{11}c$ by ab^4c^2.
7. $10x^4b^{14}c^3$ by $2xb^4c$.
8. $36x^4y^3$ by $6x^3y^5$.
9. $35x^{2a-1}y^{2x+1}$ by $5x^{a-1}y^{x+1}$.
10. $2m^{2a+4}n^{3a-2}$ by $m^{a+2}n^{a-2}$.

In each of the following exercises state which of the Principles I–XVII are used:

Divide:

11. $2^3 \cdot 3^2 - 2^4 \cdot 3^3$ by $2^3 \cdot 3^2$.
12. $5 \cdot 2^7 \cdot 3^8 + 7 \cdot 2^5 \cdot 3^4$ by $2^5 \cdot 3^4$.
13. $4x^2y^3 - 3x^3y^2$ by x^2y^2.
14. $18x^4y^4 - 12x^3y^3 + 6x^2y^2$ by $6x^2y^2$.
15. $49a^4 + 21a^3 - 7a$ by $7a$.
16. $12ax^4y^3 - 16a^2x^3y^2 + 8a^3xy$ by $4axy$.
17. $2x^{3a} + 4x^{4a} - 8x^{2a}$ by $2x^a$.
18. $6x^{2n+1} + 12x^{3n+1} - 10x^{n+1}$ by $2x^{n+1}$.
19. $4x^{13} - 6x^{11}b - 10x^4c$ by $2x^4$.
20. $10a^3b^2 - a^2b^3 + 15a^4b^4$ by $5a^2b^2$.

SQUARE ROOTS OF MONOMIALS

158. **Definition.** The **radical sign,** $\sqrt{}$, indicates that we are to find one of the two equal factors of the number expression which follows it, and the **vinculum** is attached to it, $\sqrt{}$, to show how far its effect is to extend.

E.g. $\sqrt{9}$ is read the *square root of* 9.
Similarly, $\sqrt{a^2+2ab+b^2}$ is read the *square root* of $a^2+2ab+b^2$.

The square root of any number is at once evident if we can resolve it into two equal groups of factors.

E.g.
$$\sqrt{576}=\sqrt{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot3\cdot3}=\sqrt{(2^3\cdot3)(2^3\cdot3)}=\sqrt{24\cdot24}=24.$$

It should be noted that every square has *two* square roots.

E.g. $\sqrt{9}=-3$ as well as $+3$, since $(-3)^2=9$ and $3^2=9$.

In obtaining a square root the two results should always be indicated. This is usually done by attaching the double sign $\pm$ to the square root, *e.g.* $\sqrt{9}=\pm3$.

EXERCISES

Find by inspection the following square roots:

1. $\sqrt{4}$.	8. $\sqrt{121}$.	15. $\sqrt{324}$.	22. $\sqrt{2^4}$.
2. $\sqrt{9}$.	9. $\sqrt{169}$.	16. $\sqrt{289}$.	23. $\sqrt{5^{12}}$.
3. $\sqrt{16}$.	10. $\sqrt{225}$.	17. $\sqrt{625}$.	24. $\sqrt{7^8}$.
4. $\sqrt{25}$.	11. $\sqrt{196}$.	18. $\sqrt{900}$.	25. $\sqrt{a^{12}}$.
5. $\sqrt{36}$.	12. $\sqrt{256}$.	19. $\sqrt{10000}$.	26. $\sqrt{3^{14}}$.
6. $\sqrt{49}$.	13. $\sqrt{576}$.	20. $\sqrt{a^4}$.	27. $\sqrt{a^{24}}$.
7. $\sqrt{81}$.	14. $\sqrt{400}$.	21. $\sqrt{x^6}$.	28. $\sqrt{3^4}$.

159. The square root of the product of several factors, each of which is a square, may be found in two ways if the factors are expressed in Arabic figures.

E.g. $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{1600} = \sqrt{40 \cdot 40} = \pm 40,$

or $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{2^2 \cdot 4^2 \cdot 5^2} = \pm 2 \cdot 4 \cdot 5 = \pm 40.$

But with literal factors, the second process only is available.

E.g. $\sqrt{16\, a^2b^4c^2} = \sqrt{4^2\, a^2b^4c^2} = \pm 4\, ab^2c.$

EXERCISES

Find the following indicated square roots, keeping the results in the factored form:

1. $\sqrt{2^2 \cdot 3^2}$.	7. $\sqrt{3^{12} \cdot 5^{14}}$.	13. $\sqrt{9\, x^4y^{12}}$.
2. $\sqrt{81 \cdot 121}$.	8. $\sqrt{2^{22} \cdot 3^{12}}$.	14. $\sqrt{121\, a^2x^4}$.
3. $\sqrt{49 \cdot 25 \cdot 169}$.	9. $\sqrt{16\, a^2b^2c^2}$.	15. $\sqrt{7^4\, a^4b^2}$.
4. $\sqrt{8^2 \cdot 5^2 \cdot 3^2}$.	10. $\sqrt{64\, a^4x^4}$.	16. $\sqrt{625\, x^4y^2}$.
5. $\sqrt{5^4 \cdot 3^2 \cdot 4^4}$.	11. $\sqrt{4^4\, a^2b^4}$.	17. $\sqrt{1225\, a^{2r}}$.
6. $\sqrt{25 \cdot 36}$.	12. $\sqrt{3^2\, x^2y^2}$.	18. $\sqrt{36\, b^{4m}c^{2n}}$.

Notice that the square root of a *sum* is *not* obtained by taking the square roots of the terms separately. Thus, $\sqrt{9+16}$ is not equal to $\sqrt{9}+\sqrt{16}$.

The preceding exercises illustrate the following principle:

160. Principle XVIII. *The square root of a product is obtained by finding the square root of each factor separately and then taking the product of these roots.*

In order that a factor may be a perfect square it must be a power whose exponent is *even*. Its square root is then a

power of the same base whose exponent is equal to one-half the given exponent.

Thus, $\sqrt{x^6} = \sqrt{x^3 \cdot x^3} = x^3$. The exponent 3 of the root is one-half the exponent 6 of the power. Hence to find the square root of a monomial we divide the exponent of each factor by 2.

EXERCISES

Find the following square roots:

1. $\sqrt{4\,a^6b^8}$.
2. $\sqrt{3^2x^{12}y^{14}}$.
3. $\sqrt{5^2 \cdot 3^{22}\,t^{24}}$.
4. $\sqrt{121\,x^4y^{12}}$.
5. $\sqrt{576\,a^2b^4}$.
6. $\sqrt{10^4\,a^4b^4}$.
7. $\sqrt{5^4\,m^{14}}$.
8. $\sqrt{5^4 \cdot 3^8 \cdot 7^2}$.
9. $\sqrt{3^{14} \cdot 7^{12}\,a^4}$.
10. $\sqrt{25\,a^2b^4c^{12}}$.
11. $\sqrt{81\,x^4y^8c^{10}}$.
12. $\sqrt{729\,a^6y^{10}z^{14}}$.
13. $\sqrt{64 \cdot 625\,a^2b^4}$.
14. $\sqrt{256\,x^{2k}y^{4n}}$.
15. $\sqrt{3^{2x} \cdot 5^{2y}}$.

DIVISION BY A POLYNOMIAL

161. The simplest case of division by a polynomial is that in which the dividend can be resolved into two factors, one being the given polynomial divisor and the other a monomial.

E.g. To divide $4\,x^3 + 4\,x^2y$ by $x + y$, factor the dividend and we have
$$4\,x^2\,(x+y) \div (x+y) = 4\,x^2.$$

In case the dividend cannot be factored in this manner, then, if the division is possible, the quotient must be a polynomial. The process of finding the quotient under such circumstances is best shown by studying a particular case.

Illustrative Example 1. Consider the product

$$(x^2 + 2\,xy + y^2)(x+y) = x^2(x+y) + 2\,xy\,(x+y) + y^2\,(x+y).$$

The products, $x^2(x+y)$, $2\,xy(x+y)$, and $y^2(x+y)$ are called **partial products**, and their sum, $x^3 + 3\,x^2y + 3\,xy^2 + y^3$, the **complete product.**

In dividing $x^3 + 3\,x^2y + 3\,xy^2 + y^3$ by $x + y$ the quotient must be such a polynomial that when its terms are multiplied by $x + y$ the

results are these partial products, which in the solution are called 1st, 2d, and 3d products.

The work may be arranged as follows:

Dividend or product:	$x^3 + 3x^2y + 3xy^2 + y^3$	$x + y$, divisor.
1st product, $x^2(x+y)$:	$x^3 + x^2y$	$x^2 + 2xy + y^2$,
Dividend minus 1st product:	$2x^2y + 3xy^2 + y^3$	[quotient.
2d product, $2xy(x+y)$:	$2x^2y + 2xy^2$	
Dividend minus 1st and 2d products:	$xy^2 + y^3$	
3d product, $y^2(x+y)$:	$xy^2 + y^3$	
Dividend minus 1st, 2d, and 3d products:	0	

Explanation. Since the dividend or product contains the term x^3, and since one of the factors, the divisor, contains the term x, the other factor, the quotient, must contain the term x^2. Multiplying this term of the quotient by the divisor, we obtain the first partial product, $x^3 + x^2y$.

Subtracting the first partial product from the whole product $x^3 + 3x^2y + 3xy^2 + y^3$, the remainder is $2x^2y + 3xy^2 + y^3$, which is the product of the divisor and that part of the quotient which has not yet been found. Since this product contains the term $2x^2y$ and the divisor contains the term x, the quotient must contain the term $2xy$. The product of $2xy$ and $x + y$ is the second partial product.

Subtracting this second partial product from $2x^2y + 3xy^2 + y^3$, we have $xy^2 + y^3$ still remaining after the first and second partial products have been subtracted from the whole product. This remainder is the product of the divisor and the part of the quotient not yet found. Since the product contains the term xy^2 and the divisor contains the term x, the quotient must contain the term y^2; hence, the third partial product is $xy^2 + y^3$.

Subtracting the third partial product the remainder is zero. Hence the sum of the three partial products thus obtained is equal to the whole product, and it follows that $x^2 + 2xy + y^2$ is the required quotient.

162. Problems in division may be checked by substituting any convenient values for the letters. For example, in this case, $x=1$, $y=1$, reduces the dividend to 8, the divisor to 2, and the quotient to 4, which verifies the correctness of the result.

Since division by zero is impossible (see Advanced Course), care must be taken not to select such values for the letters as will reduce the divisor to zero.

Illustrative Example 2. Divide $2x^4+x^3-7x^2+5x-1$ by x^2+2x-1.

Solution.

		[divisor.
Dividend or product:	$2x^4+x^3-7x^2+5x-1$	x^2+2x-1,
1st product, $2x^2(x^2+2x-1)$:	$2x^4+4x^3-2x^2$	$2x^2-3x+1$,
Dividend minus 1st product:	$-3x^3-5x^2+5x-1$	[quotient.
2d product, $-3x(x^2+2x-1)$:	$-3x^3-6x^2+3x$	
Dividend minus 1st and 2d products:	$+x^2+2x-1$	
3d product, $1\cdot(x^2+2x-1)$:	x^2+2x-1	
Dividend minus 1st, 2d, and 3d products:	0	

Check. Substitute $x=2$ in dividend, divisor, and quotient.

Illustrative Example 3. Divide $20a^2-8+18a^4+22a-19a^3$ by $2a^2-3a+4$.

Solution. Arranging dividend and divisor according to the descending powers of a, we have

		[divisor.
Dividend or product:	$18a^4-19a^3+20a^2+22a-8$	$2a^2-3a+4$,
1st product:	$18a^4-27a^3+36a^2$	$9a^2+4a-2$,
Dividend minus 1st product:	$8a^3-16a^2+22a-8$	[quotient.
2d product:	$8a^3-12a^2+16a$	
Dividend minus 1st and 2d products:	$-4a^2+6a-8$	
3d product:	$-4a^2+6a-8$	
Dividend minus all products:	0	

Check. Substitute $a=1$ in dividend, divisor, and quotient.

163. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor. This quotient is the first term of the quotient.

3. Multiply the first term of the quotient by the divisor and subtract the product from the dividend.

4. Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.

5. Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor. In case no remainder is zero, the division is not exact.

EXERCISES

Check the result in each case, being careful to substitute such numbers for the letters as do not make the divisor zero.

Divide the following:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $a^2 - 2ab + b^2$ by $a - b$.
3. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
4. $2x^3 + 2x^2y - 4x^2 - x - 4xy - y$ by $x + y$.
5. $x^3 + xy^2 - x^2y - y^3$ by $x - y$.
6. $x^3 + 4x^2 + x - 6$ by $x + 3$.
7. $x^3 + 4x^2 + x - 6$ by $x - 1$.
8. $x^4 - 6x^3 + 2x^2 - 3x + 6$ by $x - 1$.

9. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.

10. $x^3 - 8x^2 + 75$ by $x - 5$.

11. $2a^3 + 19a^2b + 9ab^2$ by $2a + b$.

12. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x - y$.

13. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.

14. $x^4 + x^3y + xy^3 + y^4$ by $x + y$.

15. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

16. $x^4 - y^4$ by $x - y$.

17. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

18. $2x^4 + 11x^3 - 26x^2 + 16x - 3$ by $x^2 + 7x - 3$.

19. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.

20. $x^5 - x^4 - 27x^3 + 10x^2 - 30x - 200$ by $x^2 - 4x - 10$.

21. $3x^2 - 4xy + 8xz - 4y^2 + 8yz - 3z^2$ by $x - 2y + 3z$.

22. $9r^2s^2 - 4r^2t^2 + 4rst^2 - s^2t^2$ by $3rs - 2rt + st$.

23. $9a^2b^2 + 16x^2 - 4a^2 - 36b^2x^2$ by $3ab + 6bx - 2a - 4x$.

24. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.

25. $a^5 + a^4b + a^3 - a^3b^2 - 2ab^2 + b^3$ by $a^2 + ab - b^2$.

26. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$.

27. $a^3 + b^3 + 3ab - 1$ by $a + b - 1$.

28. $6x^{5k} - 11x^{4k} + 23x^{3k} + 13x^{2k} - 3x^k + 2$ by $3x^k + 2$.

29. $a^{3k} - 3a^{2k}b^k + 3a^kb^{2k} - b^{3k}$ by $a^k - b^k$.

30. $32s^{4a} - 9s^{3a}t^b + 12s^{2a}t^{2b} - 18s^at^{3b} - 17t^{4b}$ by $s^a - t^b$.

164. Since the dividend is the product of the divisor and quotient, it follows that if one factor of an expression is given, the other factor may be found by division.

EXERCISES

Obtain the factors of each of the following products, one factor being given in each case:

	Product	*Given factor*
1.	$x^3 - y^3$.	$x - y$.
2.	$x^3 + y^3$.	$x + y$.
3.	$a^5 - b^5$.	$a - b$.
4.	$a^5 + b^5$.	$a + b$.
5.	$a^6 + b^6$.	$a^2 + b^2$.
6.	$a^9 + b^9$.	$a^3 + b^3$.
7.	$r^3 - s^3$.	$r^2 + rs + s^2$.
8.	$r^4 - s^4$.	$r^3 + r^2s + rs^2 + s^3$.
9.	$r^5 + s^5$.	$r^4 - r^3s + r^2s^2 - rs^3 + s^4$.
10.	$a^3 - 12a^2 + 27a + 40$.	$a - 5$.
11.	$x^5 - 5x^4y + 11x^3y^2$ $-14x^2y^3 + 9xy^4 - 2y^5$.	$x^2 - 3xy + 2y^2$.
12.	$x^4 + x^2y^2 + y^4$.	$x^2 - xy + y^2$.
13.	$a^3 + 5a^2 - 2a - 24$.	$a^2 + 7a + 12$.
14.	$a^5 - 5a^4b + 10a^3b^2$ $-10a^2b^3 + 5ab^4 - b^5$.	$a^2 - 2ab + b^2$.
15.	$x^5 - 5x^3y^2 - 5x^2y^3 + y^5$.	$x^2 - 3xy + y^2$.

SQUARE ROOTS OF POLYNOMIALS

165. In §§ 87, 88, we found certain trinomials which were perfect squares, namely,

$$a^2 + 2ab + b^2 = (a + b)^2, \tag{1}$$

$$a^2 - 2ab + b^2 = (a - b)^2. \tag{2}$$

Hence we know the square roots of all trinomials which are in either of these forms. These trinomial squares may be used

to discover a process for finding the square root of any polynomial which is a perfect square.

Finding a square root may be regarded as a process of division in which divisor and quotient are equal and both are to be found simultaneously.

Illustrative Example. Find the square root of $4x^2+12xy+9y^2$.

Considering the formula (1) we are to pass from the square $a^2+2\,ab+b^2$ to the square root $a+b$, and for this purpose we write $a^2+2\,ab+b^2$ in the form $a^2+b\,(2\,a+b)$ and arrange the work as follows:

Square or product,	$4\,x^2+12\,xy+9\,y^2 \,\vert\, \underline{2\,x+3\,y}$,	square root.
	$4\,x^2$	1st par'l product.
1st par'l divisor, $4\,x$	$12\,xy+9\,y^2$,	square minus 1st par'l prod.
1st compl. divisor, $4\,x+3\,y$	$\underline{12\,xy+9\,y^2}$,	2d par'l product.
	0	

Supposing that $4\,x^2$ is the a^2 of the formula, a is then $2\,x$, which is the first term of the root. Squaring $2\,x$ gives $4\,x^2$, the first partial product. Subtracting $4\,x^2$ from the total product leaves $12\,xy+9\,y^2$, which is the $b\,(2\,a+b)$ of the formula.

Since b is not yet known, we cannot find completely either of the factors of $b\,(2\,a+b)$; but since a has been found, we can get the first term of the factor $2\,a+b$, viz. $2\,a$ or $2\cdot 2\,x=4\,x$, which is the first partial divisor. Dividing $12\,xy$ by $4\,x$ we have $3y$, which is the b of the formula. Then $2\,a+b=4\,x+3\,y$ the first complete divisor.

To obtain the second partial product, $b\,(2\,a+b)$ or $12\,xy+9y^2$, we multiply $4\,x+3\,y$ by $3\,y$. On subtracting, the remainder is zero and the process ends, whence the required root is $2\,x+3\,y$.

It should be clearly understood that the sum of the first and second partial products is a square, viz., the square of $2\,x+3\,y$, because it has been constructed just as $a^2+b\,(2\,a+b)$ was formed from $a+b$.

EXERCISES

Find in the manner just described the square roots of the following trinomials:

1. $36x^2 - 84xy + 49y^2$.
2. $16a^2 - 40ab + 25b^2$.
3. $121t^2 + 264ta + 144a^2$.
4. $81m^4 + 144m^2n^2 + 64n^4$.
5. $49s^6 - 84s^3 + 36$.
6. $1 - 20x^4 + 100x^8$.

166. The process just applied to trinomials is applicable to polynomial squares having a larger number of terms, by regarding the a of the formula at each step as representing the part of the root already found and b as the term of the root about to be found.

Illustrative Example. Find the square root of $x^2 + 2xy + y^2 + 6x + 6y + 9$. The work may be arranged as follows:

Square or product,		$x^2+2xy+y^2+6x+6y+9$	$x+y+3$,	square root.
		x^2		1st par'l product.
1st par'l div'r,	$2x$	$2xy+y^2+6x+6y+9$,		1st remainder.
1st compl. div'r,	$2x+y$	$2xy+y^2$,		2d par'l product.
2d par'l div'r,	$2x+2y$	$6x+6y+9$,		2d remainder.
2d compl. div'r,	$2x+2y+3$	$6x+6y+9$,		3d par'l product.
		0		

After the first two terms of the root have been found, namely, x and y, then we consider $x+y$ as the a of the formula and call it a'. The new b, which we call b', is then found to be 3. Subtracting the first and second partial products is the same as subtracting $(x+y)^2$, that is, the square of a'. Hence, the second partial divisor, which is twice a', is $2(x+y)$.

In case there are four terms in the root, then the sum of the first three, when found as above, is regarded as the new a, called a''. The remaining term of the root is the new b, and is called b''.

The formula $(a-b)^2 = a^2 - 2ab + b^2$ is not needed, since this may be written $(a+(-b))^2 = a^2 + 2a(-b) + (-b)^2$, which is in the form of $(a+b)^2 = a^2 + 2ab + b^2$.

EXERCISES

Find the square roots of the following polynomials:

1. $x^2 + y^2 + z^2 - 2\,xy + 2\,xz - 2\,yz.$
2. $9\,x^2 + 4\,y^2 + z^2 - 12\,xy + 6\,xz - 4\,yz.$
3. $a^2 - 8\,ab + 16\,b^2 - 2\,ac + c^2 + 8\,bc.$
4. $a^4 + 4\,a^3 + 6\,a^2 + 4\,a + 1.$
5. $c^4 - 4\,c^3 + 6\,c^2 - 4\,c + 1.$
6. $y^4 - 4\,y^2 - 8\,xy^2 + 16\,x + 16x^2 + 4.$
7. $x^4 - 2\,x^3 + 5\,x^2 - 4\,x + 4.$
8. $a^2 + a^2b^2 - 2\,a^2b + 2\,abc - 2\,ab^2c + b^2c^2.$
9. $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$
10. $x^4 - 4\,x^3y + 6\,x^2y^2 - 4\,xy^3 + y^4.$
11. $x^6 - 4\,x^5 + 4\,x^4 + 6\,x^3 - 12\,x^2 + 9.$
12. $a^4 + 53\,a^2 + 14\,a^3 + 28\,a + 4.$
13. $x^2 + 16\,x^2y^2 + 289 + 8\,x^2y + 34\,x + 136\,xy.$
14. $9\,a^4 + 4\,a^2 + 256 - 12\,a^3 - 96\,a^2 + 64\,a.$
15. $a^6 + 6\,a^5 + 15\,a^4 + 20\,a^3 + 15\,a^2 + 6\,a + 1.$
16. $a^6 - 6\,a^5 + 15\,a^4 - 20\,a^3 + 15\,a^2 - 6\,a + 1.$
17. $16\,x^6 + 4\,y^2 + 1 - 16\,x^3y + 8\,x^3 - 4\,y.$
18. $25 + 49\,x^2 + 4\,x^4 - 70\,x - 20\,x^2 + 28\,x^3.$
19. $64\,x^6 + 192\,x^5 + 240\,x^4 + 160\,x^3 + 60\,x^2 + 12\,x + 1.$
20. $4\,x^6 - 12\,x^5 + 13\,x^4 - 14\,x^3 + 13\,x^2 - 4\,x + 4.$
21. $a^4b^4 - 2\,a^3b^2 + a^2 - 2\,a^2b^3 + 2\,ab + b^2.$
22. $16\,a^6 + 24\,a^5 + 25\,a^4 + 20\,a^3 + 10\,a^2 + 4\,a + 1.$
23. $x^6y^6 + 2\,x^5y^5 + 3\,x^4y^4 + 4\,x^3y^3 + 3\,x^2y^2 + 2\,xy + 1.$
24. $1 + 2\,x + 3\,x^2 + 4\,x^3 + 5\,x^4 + 4\,x^5 + 3\,x^6 + 2\,x^7 + x^8.$
25. $9\,a^2 - 6\,ab + 30\,ac + 6\,ad + b^2 - 10\,bc - 2\,bd + 25\,c^2$
    $+ 10\,cd + d^2.$

SQUARE ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

167. The square root of a number expressed in Arabic figures may be found by the process just used for polynomials.

Illustrative Example. Find the square root of 5329.

In order to decide how many digits there are in the root we observe that $10^2 = 100$ and $100^2 = 10000$; hence the root lies between 10 and 100, *i.e.* it contains two digits. Since $80^2 = 6400$ and $70^2 = 4900$, we see that 7 is the largest number possible in tens' place of the root. The work is arranged as follows:

The given square,	5329	$\underline{\vert\, 70 + 3}$, square root.
$a^2 = 70^2$	$\underline{4900}$,	1st partial product.
$2\,a = 2 \times 70 = 140$	429,	1st remainder.
$b = 3$		
$2\,a + b = 143$	$\underline{429} = b\,(2\,a + b)$.	
	0	

Having decided as above that the a of the formula is 7 tens, we square this and subtract, obtaining 429 as the remaining part of the power.

The first partial divisor, $2\,a = 140$, is divided into 429 giving a quotient 3, which is the b of the formula. Hence the first complete divisor, $2\,a + b$, is 143, and the second partial product, $b\,(2\,a + b)$, is 429. Since the remainder is zero, the process is exact and 73 is the square root sought.

EXERCISES

Find the square roots of the following:

1. 3249.	5. 6241.	9. 1849.
2. 8836.	6. 7056.	10. 7225.
3. 7569.	7. 9409.	11. 3025.
4. 8281.	8. 9801.	12. 9216.

168. The square of any integer from 1 to 9 contains one or two digits; the square of any integer from 10 to 99 contains three or four digits; the square of any integer from 100 to 999 contains five or six digits; etc.

Hence it is evident that, if the digits of a number which is a perfect square be separated into groups of two each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the square root.

Illustrative Example. Find the square root of 120,409.

Since this number is divided into three groups, the first digit is in hundreds' place. The work is arranged as follows:

Square,	12 04 09	$300 + 40 + 7 = 347$, square root.
$a^2 = 300^2$	9 00 00,	1st partial product.
$2a = 2 \times 300 = 600$	3 04 09,	1st remainder.
$b = 40$		
$2a + b = 640$	2 56 00 $= b(2a + b)$.	
$2a' = 2 \times 340 = 680$	48 09,	2d remainder.
$b' = 7$		
$2a' + b' = 687$	48 09 $= b'(2a' + b')$.	
	0	

The first partial divisor, 2×300, is completed by adding the second term of the root, 40, that is, $2a + b = 600 + 40$. At first glance it might be supposed that the second digit is 5 instead of 4, since 600 is contained in 30,409 50 times, but account must be taken of the addition to be made to the partial divisor, and when this is done the quotient is 40, not 50.

In the second partial divisor, $2a'$ stands for 2 times $(300 + 40) = 680$, and b' stands for the third digit of the root.

In case a square contains an odd number of digits the last group at the left will have one instead of two digits.

E.g. 3 47 21 has three places in its square root, of which the first, hundreds' digit, is the largest square in 3, namely, 1.

EXERCISES

Find the square root of each of the following:

1. 294,849.
2. 37,636.
3. 872,356.
4. 599,076.
5. 3481.
6. 7569.
7. 1849.
8. 73,441.
9. 100,489.
10. 26,569.
11. 874,225.
12. 170,569.
13. 35,721.
14. 16,641.
15. 32,761.
16. 223,729.

169. Since the square of any decimal fraction has twice as many places as the given decimal, it is evident that the square root of a decimal fraction contains one decimal place for every two in the square.

E.g. $(.15)^2 = .0225$; $(.012)^2 = .000144$.

Hence, for the purpose of determining the decimal places in the root, the decimal part of a square should be divided into groups of two digits each, counting from the decimal point toward the right.

Illustrative Example. Find the square root of 4.6225. According to §§ 168, 169 the root contains one digit in the integral part and two in the decimal part. The work is as follows:

$$
\begin{array}{rr|l}
 & & 4.62\ 25 \,\underline{\mid\ 2 + .1 + .05 = 2.15} \\
a^2 = 2^2 & & 4 \\
2a = 2 \times 2 = & 4.0 & .62\ 25, \quad \text{first remainder.} \\
b = & \underline{.1} & \\
2a + b = & 4.1 & \underline{.41} \quad = b\,(2a + b). \\
2a' = 2 \times 2.1 = & 4.2 & .21\ 25, \quad \text{second remainder.} \\
b' = & \underline{.05} & \\
2a' + b' = & 4.25 & \underline{.21\ 25} = b'\,(2a' + b'). \\
 & & \quad 0
\end{array}
$$

This process is also applicable for the purpose of approximating the square root of a number which is not a perfect square.

Illustrative Example. Find the approximate square root of 582 to three decimal places. The solution below shows all the steps of the work.

		$5\ 82$	$20 + 4 + .1 + .02 + .004 = 24.124$
$a^2 = 20^2$		$4\ 00$	
$2a = 2 \times 20$	$= 40.$	$1\ 82,$	first remainder.
$b =$	$4.$		
$2a + b$	$= 44.$	$1\ 76$	$= b(2a + b).$
		$6.00,$	second remainder.
$2a' = 2 \times 24$	$= 48.$		
$b' =$	$.1$		
$2a' + b' =$	48.1	4.81	$= b'(2a' + b').$
		$1.1900,$	third remainder.
$2a'' = 2 \times 24.1 = 48.2$			
$b'' =$	$.02$		
$2a'' + b'' =$	48.22	$.9644$	$= b''(2a'' + b'').$
		$.225600,$	fourth remainder.
$2a''' = 2 \times 24.12 = 48.24$			
$b''' =$	$.004$		
$2a''' + b''' =$	48.244	$.192976$	$= b'''(2a''' + b''').$
		$.032624$	

The decimal points are handled exactly as in division of decimals in arithmetic, the chief care being needed in forming the divisors.

170. Evidently the process in this example may be carried on indefinitely. 24.124 is an **approximation** to the square root of 582. In fact, the square of 24.124 differs from 582 by the small fraction .032624. 24.12 is the nearest approximation using two decimal places. If the third figure were 5 or any digit greater than 5, then 24.13 would be the nearest approximation using two decimal places. Hence three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Find the square roots of the following, correct to two decimal places:

1. 387.	7. 2.	13. .02.
2. 5276.	8. 3.	14. .003.
3. 2.92.	9. 5.	15. .5.
4. 27.29.	10. 7.	16. .005.
5. 51.	11. 8.	17. .307.
6. 3.824.	12. 11.	18. 200.002.

SQUARE ROOTS OF FRACTIONS EXPRESSED IN ARABIC FIGURES

171. Since in arithmetic the product of two fractions is found by multiplying their numerators and their denominators, a fraction is squared by squaring its numerator and its denominator separately.

Hence, to extract the square root of a fraction, we find the square root of its numerator and its denominator separately.

E.g. $\sqrt{\frac{16}{25}} = \frac{4}{5}$, since $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

However, in approximating the square root of a fraction whose denominator is not a perfect square, if the final result is to be reduced to a decimal, the direct use of this method is cumbersome since it usually involves the approximation of two square roots and always necessitates dividing by a long decimal fraction.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{2} \div \sqrt{3} = 1.4142 \div 1.7321$, in which we are now obliged to divide by 1.7321.

This difficulty may be avoided in either of two ways:

1. The fraction may be reduced to a decimal before the root is approximated.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{.666\cdots} = .8165.$

2. The denominator of the fraction may be made a perfect square before approximating the root.

E.g. $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\frac{\sqrt{6}}{3}=\frac{2.4495}{3}=.8165.$

By either of these two processes only one square root is approximated, and there is only a short division by 3 instead of a long division by 1.7321.

It is clear that any fraction can be changed into an equal fraction whose denominator is a perfect square by multiplying numerator and denominator by the proper number.

E.g. $\frac{1}{2}=\frac{2}{4}$, $\frac{3}{5}=\frac{15}{25}$, $\frac{7}{8}=\frac{14}{16}$, etc. If a fraction is given in the form $\frac{1}{\sqrt{3}}$, it may be written $\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{9}}=\frac{1}{3}\sqrt{3}$. In like manner,

$$\frac{7}{\sqrt{11}}=7\cdot\frac{1}{\sqrt{11}}=7\sqrt{\frac{1}{11}}=7\sqrt{\frac{11}{121}}=\frac{7}{11}\sqrt{11}.$$

EXERCISES

Find approximately correct to two decimal places the following square roots.

In the first ten obtain the results in three different ways: (*a*) Find the root of each numerator and denominator separately; (*b*) reduce each fraction to a decimal; (*c*) reduce each fraction so as to make its denominator a perfect square.

In the remaining exercises use methods (*b*) and (*c*) only. In each case compare the results obtained.

1. $\sqrt{\frac{3}{5}}$.
2. $\sqrt{\frac{4}{8}}$.
3. $\sqrt{\frac{5}{7}}$.
4. $\sqrt{\frac{11}{18}}$.
5. $\sqrt{\frac{5}{8}}$.
6. $\sqrt{\frac{4}{5}}$.
7. $\sqrt{\frac{31}{8}}$.
8. $\sqrt{\frac{42}{51}}$.
9. $\sqrt{\frac{6}{7}}$.
10. $\sqrt{\frac{5}{6}}$.
11. $\sqrt{\frac{25}{4}}$.
12. $\sqrt{\frac{16}{7}}$.
13. $\sqrt{\frac{1}{2}}$.
14. $\sqrt{\frac{3}{7}}$.
15. $\sqrt{\frac{1}{7}}$.
16. $\sqrt{\frac{2}{5}}$.
17. $\frac{1}{\sqrt{5}}$.
18. $\frac{5}{\sqrt{13}}$.
19. $\frac{3}{\sqrt{7}}$.
20. $\frac{7}{\sqrt{17}}$.
21. $\sqrt{\frac{3}{8}}$.

172. Principle XVIII may be used to advantage in approximating the square roots of certain integral numbers.

E.g. suppose $\sqrt{2}$ has been computed, and $\sqrt{8}$ is desired. It is unnecessary to compute the $\sqrt{8}$ directly, for by XVIII,

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$$

This sort of simplification is possible whenever the number under the radical sign can be resolved into two factors, one of which is a perfect square.

E.g. suppose the $\sqrt{5}$ to have been computed, then

$$\sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}.$$

In like manner, $\sqrt{a^5b^3}$ may be written

$$\sqrt{a^4b^2 \cdot ab} = \sqrt{a^4b^2} \cdot \sqrt{ab} = a^2b\sqrt{ab}.$$

Definition. A radical expression is said to be **simplified** when the number under the radical sign is in the integral form and contains no factor which is a perfect square.

E.g. the simplified forms of $\sqrt{125}$, $\sqrt{a^5b^3}$, $\sqrt{\frac{1}{5}}$, $\dfrac{1}{\sqrt{3}}$, are respectively,

$$5\sqrt{5},\ a^2b\sqrt{ab},\ \tfrac{1}{5}\sqrt{5},\ \tfrac{1}{3}\sqrt{3}.$$

EXERCISES

Given $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7321$, $\sqrt{5} = 2.2361$, compute the following, correct to three places of decimals, without further extraction of roots:

1. $\sqrt{80}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{48}$.
5. $\sqrt{75}$.
6. $\sqrt{2 \cdot 3}$.
7. $\sqrt{72}$.
8. $\sqrt{98}$.
9. $\sqrt{363}$.
10. $\sqrt{125}$.
11. $\sqrt{27} + \sqrt{\frac{1}{3}}$.
12. $\sqrt{45} + \sqrt{\frac{1}{5}}$.
13. $\sqrt{50} - \sqrt{\frac{1}{2}} + \sqrt{8}$.
14. $\sqrt{48} + \sqrt{75} - \sqrt{3}$.
15. $\sqrt{32} + \sqrt{72} - \sqrt{18}$.

Simplify the following:

16. $\sqrt{32\,a^2b}$.
19. $\sqrt{45x^3y^5b^3}$.
22. $\sqrt{500\,x^7\,a^3b}$.
17. $\sqrt{81\,x^3b^2}$.
20. $\sqrt{63\,bc^5d^4}$.
23. $\sqrt{3\,x^2+6\,xy+3\,y^2}$.
18. $\sqrt{50\,a^5b^4c^2}$.
21. $\sqrt{900\,ab^4c^5}$.
24. $\sqrt{8\,x^2-12\,y^2}$.
25. $\sqrt{32\,a^2-64\,ab+32\,b^2}$.
26. $\sqrt{125\,x^2+250\,xy+125\,y^2}$.

27. Find approximately to four decimal places the sides of a square whose area is 120.

28. Approximate to four decimals the side of a square having an area equal to that of a rectangle whose sides are 15 and 20.

29. How many rods of fence are required to fence a square piece of land containing 50 acres, each acre containing 160 square rods?

30. A square checkerboard has an area of 324 square inches. What are its dimensions?

173. In adding or subtracting expressions containing radicals it is always best to first reduce each radical expression to its simplest form, since this often gives opportunity to combine terms which are similar with respect to some radical expression.

Ex. 1. $\sqrt{32}+\sqrt{72}-\sqrt{18}=4\sqrt{2}+6\sqrt{2}-3\sqrt{2}=7\sqrt{2}$ by Principles XVIII, I, and II.

Ex. 2. $\sqrt{\tfrac{1}{3}}+\sqrt{12}-\sqrt{\tfrac{3}{4}}=\tfrac{1}{3}\sqrt{3}+2\sqrt{3}-\tfrac{1}{2}\sqrt{3}$
$=(\tfrac{1}{3}+2-\tfrac{1}{2})\sqrt{3}=1\tfrac{5}{6}\sqrt{3}$.

EXERCISES

Simplify each of the following as far as possible without approximating roots.

1. $\sqrt{27}+2\sqrt{48}-3\sqrt{75}$.
2. $\sqrt{20}+\sqrt{125}-\sqrt{180}$.
3. $3\sqrt{432}-4\sqrt{3}+\sqrt{147}$.

4. $3\sqrt{2450} - 25\sqrt{2} + 4\sqrt{13122}$.

5. $3y^2\sqrt{x^3z} + 2\sqrt{x^5z^3} - yz^4\sqrt{\frac{xz}{y^2}}$.

6. $\sqrt{4x^3y} + \sqrt{25xy^3} - x\sqrt{xy}$.

7. $\sqrt{ax^2 - bx^2} + \sqrt{4ar^2s^2 - 4br^2s^2}$.

8. $4\sqrt{\frac{3}{4}} - \frac{3}{4}\sqrt{\frac{3}{16}} - 3\sqrt{27}$.

9. $2\sqrt{\frac{5}{3}} + \sqrt{60} + \sqrt{\frac{3}{5}}$.

10. $5\sqrt{3} - 2\sqrt{48} + 7\sqrt{108}$.

11. $\sqrt{a^3 - a^2b} - \sqrt{ab^2 - b^3} - \sqrt{(a+b)(a^2 - b^2)}$.

12. $\sqrt{a} + 3\sqrt{2a} - 2\sqrt{3a} + \sqrt{4a} - \sqrt{8a} + \sqrt{12a}$.

13. $\sqrt{x^3 + 2x^2y + xy^2} - \sqrt{x^3 - 2x^2y + xy^2} - \sqrt{4xy^2}$.

14. $\sqrt{r - s} + \sqrt{16r - 16s} + \sqrt{rt^2 - st^2} - \sqrt{9(r - s)}$.

15. $\sqrt{(m - n)^2a} + \sqrt{(m + n)^2a} - \sqrt{am^2} + \sqrt{a(1 - m)^2} - \sqrt{a}$.

16. $\sqrt{32x^2y^4} + \sqrt{162x^2y^4} - \sqrt{512x^2y^4} + \sqrt{1250x^2y^4}$.

APPLICATIONS OF SQUARE ROOT

174. Some of the most interesting and useful applications of the square root process are concerned with the sides and areas of triangles.

The fact that the sum of the squares on the two sides of a right triangle equals the square on the hypotenuse was used in Chapter VI. (Pythagorean Proposition, page 206.)

If a and b are the lengths of the sides, and c the length of the hypotenuse, all measured in the same unit, this proposition says:

$$c^2 = a^2 + b^2. \tag{1}$$

Hence, by S,

$$a^2 = c^2 - b^2, \tag{2}$$

and

$$b^2 = c^2 - a^2. \tag{3}$$

Taking the square root of both sides in each of these equations,

$$c = \sqrt{a^2 + b^2}. \tag{4}$$

$$a = \sqrt{c^2 - b^2}. \tag{5}$$

$$b = \sqrt{c^2 - a^2}. \tag{6}$$

The negative square root is omitted here, as a negative length cannot apply to the side of a triangle. By these formulas, if any two sides of a right triangle are given, the other may be found.

E.g. if $a = 4, b = 3$, then, by (4),

$$c = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

If $c = 5, b = 3$, then, by (5),

$$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$$

If $c = 5, a = 4$, then, by (6),

$$b = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

Illustrative Problem. If the two sides of a right triangle are 8 and 12, find the hypotenuse correct to two decimal places.

Solution. We have $c = \sqrt{a^2 + b^2} = \sqrt{64 + 144} = \sqrt{208}$,

$$\sqrt{208} = \sqrt{16 \cdot 13} = \sqrt{16} \cdot \sqrt{13} = 4\sqrt{13} = 4(3.605) = 14.420.$$

PROBLEMS

In solving the following problems, simplify each expression under the radical sign before extracting the root. Find all results correct to two decimal places. In each case construct a figure.

1. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

2. The hypotenuse of a right triangle is 9 inches and one of the sides is 6 inches. Find the other side.

3. The hypotenuse of a right triangle is 25 feet and one of the sides is 15 feet. Find the other side.

4. The hypotenuse of a right triangle is 7 rods and one of the sides is 5 rods. Find the other side.

5. The hypotenuse of a right triangle is 12 inches and the two sides are equal. Find their length.

Let s equal the length of one of the equal sides.

Then $$s^2 + s^2 = 144.$$

$$2s^2 = 144.$$

$$s^2 = 72.$$

$$s = \sqrt{72} = 6\sqrt{2} = 6 \times 1.414 = 8.484.$$

6. The hypotenuse of a right triangle is 30 feet and the sides are equal. Find their length.

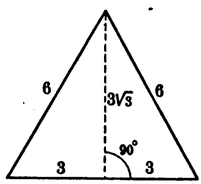
7. The hypotenuse of a right triangle is h and the sides are equal. Find their length. Solve 5 and 6 by means of the formula here obtained.

8. The diagonal of a square is 8 feet. Find its area.

9. The diagonal of a square is d. Find an expression in terms of d representing its area.

10. The side of an equilateral triangle is 6 inches. Find the altitude.

A line drawn from a vertex of an equilateral triangle perpendicular to the base meets the base at its middle point. Hence this problem becomes: the hypotenuse of a right triangle is 6 and one side is 3. Find the remaining side.



11. The side of an equilateral triangle is 10. Find the altitude.

12. The side of an equilateral triangle is s. Find the altitude.

This is equivalent to finding a side of a right triangle whose hypotenuse is s, the other side being $\frac{s}{2}$. Let h equal altitude.

Then
$$h = \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}}$$
$$= \sqrt{\frac{4s^2 - s^2}{4}} = \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{s^2}{4} \cdot 3}$$
$$= \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} = \frac{s}{2}\sqrt{3}.$$

This formula gives the altitude of any equilateral triangle in terms of the side. By means of this formula solve 11 and 12. It is interesting to notice that the square root of 3 is the only root required in finding the altitude of any equilateral triangle whatever.

13. Find the altitude of an equilateral triangle whose side is $4\frac{1}{2}$. Substitute in the formula under 12.

14. Find the area of an equilateral triangle whose side is 5.

Since the area of a triangle is $\frac{1}{2}$ the product of the base and altitude, we first find the altitude by means of the formula under 12, and then multiply by $\frac{1}{2}$ the base.

15. Find the area of the equilateral triangle whose side is s. Show the result to be $\frac{s^2}{4}\sqrt{3}$.

16. If the area of an equilateral triangle is 16 square inches, find the length of the side.

Let s equal the length of the side. Then by the formula derived under 15, $16 = \frac{s^2}{4}\sqrt{3}$.

Hence (§§ 171, 172), $s^2 = \frac{64}{\sqrt{3}} = \frac{64}{3}\sqrt{3} = 21.33 \times 1.732.$

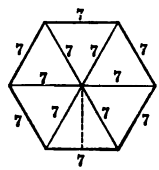
17. The area of an equilateral triangle is 50 square inches. Find its side and altitude.

18. The area of an equilateral triangle is a square inches. Find the side.

Solve the equation $a = \frac{s^2}{4}\sqrt{3}$ for s, and simplify the expression, finding $s^2 = \frac{4a}{\sqrt{3}}$, and $s = \sqrt{\frac{4a\sqrt{3}}{3}} = \frac{2}{3}\sqrt{3a\sqrt{3}}.$

19. The area of an equilateral triangle is 240 square inches. Find its side. (Substitute in the formula obtained under 18).

20. Find the area of a regular hexagon whose side is 7.



A regular hexagon is composed of 6 equal equilateral triangles, whose sides are each equal to the side of the hexagon (see figure). Hence this problem may be solved by finding the area of an equilateral triangle whose side is 7, and multiplying the result by 6.

21. Find the area of a regular hexagon whose side is s. (Solve 20 by substituting in the formula obtained here.)

22. The area of a regular hexagon is 108 square inches. Find its side.

If the area of the hexagon is 108 square inches, the area of one of the equilateral triangles is 18 square inches. Hence this problem can be solved like 18.

23. The area of a regular hexagon is a square inches. Find its side. (Solve 22 by substituting in the formula obtained here.)

24. Find the radius of a circle whose area is 9 square inches.

The area of a circle is found by squaring the radius and multiplying by 3.1416. The number 3.1416 is approximately the quotient obtained by dividing the length of the circumference by the diameter of the circle. This quotient is represented by the Greek letter π

(pronounced pi). In this chapter we use $3\frac{1}{7}$ as an approximation to π. This differs from the real value of π by less than .0013, and hence is accurate enough for most purposes. If a represents the area of a circle, the above rule may be written

$$a = \pi r^2.$$

Hence if $a = 9$, $\quad r^2 = \frac{9}{\pi} = \frac{9}{3\frac{1}{7}} = \frac{63}{22} = 2.863,$

and $\quad r = \sqrt{2.863}.$

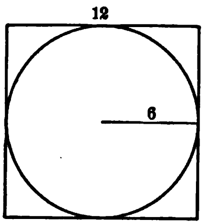
25. Find the radius of a circle whose area is 68 square feet.

26. Find the radius of a circle whose area is a square feet.

We have $\quad a = \pi r^2$, or $r^2 = \frac{a}{\pi}$,

Hence $\quad r = \sqrt{\frac{a}{\pi}} = \sqrt{\frac{a\pi}{\pi^2}} = \frac{1}{\pi}\sqrt{a\pi}.$

In problems stated in terms of letters, the results, of course, cannot be reduced to a decimal. In such formulas it is best not to replace the letter π by any of its approximations.



27. Find the sum of the areas of a circle of radius 6 and the square circumscribed about the circle.

The area of the circle is $6^2\pi = 36\,\pi$, and the area of the square is $4 \cdot 6^2 = 4 \cdot 36$; *i.e.* the square contains 4 squares whose sides are 6. The sum of the areas is

$$4 \cdot 36 + 36\,\pi = (4 + \pi)\,36 = (4 + 3\tfrac{1}{7})\,36.$$

28. Find an expression for the sum of the areas of a circle of radius r and the circumscribed square. (Solve 27 by substituting in the formula here obtained.)

29. If the sum of the areas of a circle and the circumscribed square is 64, find the radius of the circle.

By the formula obtained under 28,

$$64 = (4 + \pi)\,r^2 = \tfrac{50}{7}r^2.$$

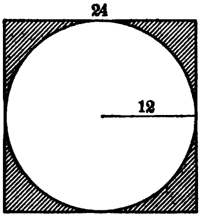
$$r^2 = 64 \cdot \tfrac{7}{50} = 8.96,$$

Hence, $\quad r = \sqrt{8.96} = 2.99.$

30. If the sum of the areas of a circle and the circumscribed square is 640 square feet, find the radius of the circle.

31. The sum of the areas of a circle and the circumscribed square is a. Find an expression representing the radius of the circle. (Replace π by $3\frac{1}{7}$ before simplifying.)

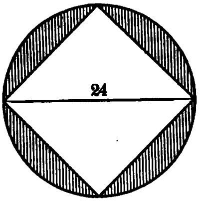
32. If the radius of a circle is 12, find the difference between the area of a circle and the circumscribed square.



33. If the radius of a circle is r, find the difference between the area of the circle and the circumscribed square. (Solve 32 by substituting in the formula obtained here.)

34. If the radius of a circle is 16, find the area of the inscribed square. (This is the same problem as finding the area of a square whose diagonal is 32. See problems 8 and 9.)

35. If the radius of a circle is r, find an expression representing the area of the inscribed square. (This is problem 9, the hypotenuse being $2r$.)

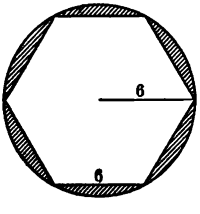


36. If the radius of a circle is 12, find the difference between the area of the circle and the area of the inscribed square.

37. If the radius of a circle is r, find an expression representing the difference between the areas of the circle and the inscribed square.

38. The radius of a circle is 10. Find the area of an inscribed hexagon. See note, problem 20.

39. The radius of a circle is 6. Find the difference between the areas of the circle and the inscribed hexagon.



40. Find an expression representing the difference between the areas of a circle with radius r and the inscribed regular hexagon.

SOLUTION OF QUADRATIC EQUATIONS BY MEANS OF SQUARE ROOT

175. **Illustrative Problem.** A rectangular field is 18 rods longer than it is wide and its area is 50 acres. What are its dimensions?

Let	$x =$ the width of the field.	
Then,	$x + 18 =$ the length of the field.	
And	$x(x+18) =$ the number of square rods in the field.	
Hence,	$x^2 + 18x = 8000,$	(1)
or	$x^2 + 18x - 8000 = 0.$	(2)

We are not able to factor the left member of the equation by any method thus far known, and hence we cannot solve the equation as in § 145. We therefore proceed to study a *general method* of solving quadratic equations.

Consider the equation in the form (1) above. We seek a number k^2 such that $x^2 + 18x + k^2$ shall be a perfect square. By § 134 the middle term of a trinomial square is twice the product of the square roots of the two square terms. Hence, $18x = 2kx$, that is, k must equal 9.

Hence, adding $9^2 (= k^2)$ to each member of (1),

$$x^2 + 18x + 9^2 = 8000 + 9^2, \tag{3}$$

or,

$$x^2 + 18x + 81 = 8081. \tag{4}$$

Since the left member is now a perfect square, we may extract the square root of both sides, approximating the root on the right.

Hence, $\quad x + 9 = \pm\sqrt{8081} = \pm 89.89,$

giving $\quad x = -9 + 89.89 = 80.89,$

and also, $\quad x = -9 - 89.89 = -98.89.$

In this case the negative result is not applicable to the problem. Hence the width of the field is 80.89 rods, which is correct to two decimal places.

Illustrative Example. Solve the equation:

$$x^2 - 12x + 42 = 56. \qquad (1)$$

By S, $\quad x^2 - 12x = 14. \qquad (2)$

By A, $\quad x^2 - 12x + k^2 = 14 + k^2. \qquad (3)$

By § 175, $\quad -12x = 2kx$ or $k = -6.$

Hence, $\quad x^2 - 12x + (-6)^2 = 14 + 36 = 50. \qquad (4)$

Taking square roots, $\quad x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}. \qquad (5)$

By A, $\quad x = 6 \pm 7.071. \qquad (6)$

Hence $\quad x = 6 + 7.071 = 13.071,$

and also $\quad x = 6 - 7.071 = -1.071.$

176. This process is called solving the quadratic equation by **completing the square**, since in each case a number is added to both sides which makes the left member a trinomial square.

Since the process always involves extracting the square root in order to find the value of the unknown, the two solutions of a quadratic equation are commonly called the **roots of the equation**. By analogy the solutions of any equation are sometimes called its roots.

EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. (§ 172.) Use these square roots only.

1. $x^2 - 4x = 8.$
2. $x^2 = 3 - 6x.$
3. $4x = 16 - x^2.$
4. $x^2 + 6x = 9.$
5. $x^2 + 6x = 11.$
6. $x^2 - 12x = 12.$
7. $x^2 - 8x = -14.$
8. $x^2 = 2x + 1.$
9. $x^2 - 4x = 16.$
10. $10x^2 = 14 + 4x.$
11. $24 = 3x^2 + 12x.$
12. $69 - 18x = 3x^2.$
13. $84 + 24x = 12x^2.$
14. $25 - x^2 = 5x.$
15. $x^2 + \frac{7}{3}x = 2.$
16. $x^2 - \frac{3}{2}x = \frac{27}{2}.$

177. In case the coefficient of x^2 is not unity, both members may be divided by this coefficient.

EXAMPLE. Solve $3x^2 + 8x = 4$. (1)

By D, $x^2 + \frac{8}{3}x = \frac{4}{3}$. (2)

By A, $x^2 + \frac{8}{3}x + k^2 = \frac{4}{3} + k^2$. (3)

By § 175, $\frac{8}{3}x = 2kx$ or $k = \frac{4}{3}$.

Hence, $x^2 + \frac{8}{3}x + (\frac{4}{3})^2 = \frac{4}{3} + \frac{16}{9} = \frac{28}{9}$. (4)

Taking square roots, $x + \frac{4}{3} = \pm\sqrt{\frac{28}{9}}$. (5)

Hence, $x = -\frac{4}{3} \pm \frac{2}{3}\sqrt{7}$, (6)

and the two roots are $x = 0.43$,

and $x = -3.10$.

The preceding example may also be solved as follows:

Multiplying each member of (1) by $4 \cdot 3 = 12$,

then, $36x^2 + 96x = 48$.

By A, $36x^2 + 96x + k^2 = 48 + k^2$,

where $12kx = 96x$ or $k = 8$.

Hence, $36x^2 + 96x + (8)^2 = 48 + 64 = 112$,

and $6x + 8 = \pm\sqrt{112} = \pm 4\sqrt{7}$.

Therefore $x = -\frac{4}{3} \pm \frac{2}{3}\sqrt{7}$.

178. The advantage of this solution is that fractions are avoided until the last step, and the value of k is found to be the *same as the coefficient of* x in the given equation. This may always be accomplished by multiplying the members of the given equation by *four times the coefficient of* x^2.

In the solution of the following equations only the square roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, need be used. In all cases where the roots are integers or exact fractions the solution may also be obtained by factoring as in § 145.

1. $2x^2 + 3x = 2$.	4. $6x + 1 = -3x^2$.	7. $2x^2 - 3x = 14$.
2. $3x^2 + 5x = 2$.	5. $2x^2 = 5x + 3$.	8. $3x^2 = 9 + 2x$.
3. $3x = 9 - 2x^2$.	6. $4x = 2x^2 - 1$.	9. $4x^2 = 2x + 1$.

10. $6x-1=3x^2$.
11. $2x^2+4x=23$.
12. $3x^2-7=4x$.
13. $2x^2-5=3x$.
14. $4x^2=6x-1$.
15. $2x=1-5x^2$.
16. $3x-20=-2x^2$.
17. $2x+3x^2=9$.
18. $4x^2-1=3x$.
19. $4x=7-2x^2$.
20. $2x+1=5x^2$.
21. $3x^2+4x=7$.
22. $3x+9=2x^2$.
23. $2x-1=-4x^2$.
24. $5x^2+16x=-2$.
25. $4x^2+1=8x$.
26. $2x^2-3x=20$.
27. $2x^2-3=-5x$.
28. $3x^2+4x=8$.
29. $10-4x=5x^2$.
30. $1+4x^2=-6x$.
31. $5-3x=2x^2$.
32. $7+4x=2x^2$.
33. $6x^2+12x=2$.
34. $6x^2-12x=-2$.
35. $6x^2+12x=-2$.
36. $6x^2-12x=2$.
37. $3x^2+2x=5$.
38. $2+3x=2x^2$.
39. $8x+1=-4x^2$.
40. $8+4x=3x^2$.
41. $10+4x=5x^2$.
42. $2+5x=3x^2$.
43. $3x+14=2x^2$.
44. $3x^2-2x=5$.
45. $2x^2+4x=1$.
46. $3x-1=-4x^2$.
47. $23+4x=2x^2$.
48. $3x-1=2x^2$.

179. Solution by Formula. Solve the equation

$$ax^2+bx+c=0. \qquad (1)$$

By S, M, $$4a^2x^2+4abx=-4ac. \qquad (2)$$

By A, $$4a^2x^2+4abx+k^2=-4ac+k^2, \qquad (3)$$

where $$4akx=4abx \text{ or } k=b.$$

Hence $$4a^2x^2+4abx+b^2=b^2-4ac. \qquad (4)$$

Taking square roots, $$2ax+b=\pm\sqrt{b^2-4ac}. \qquad (5)$$

By S, D, $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \qquad (6)$$

Calling the two values of x in the result x_1 and x_2 we have,

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a};\quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Any quadratic equation may be reduced to the form of (1) by simplifying and collecting the coefficients of x^2 and x. Hence any quadratic equation may be solved by substituting in the formulas just obtained.

Ex. 1. Solve $x^2 - 4x + 1 = 0.$

In this case $a = 1, b = -4, c = 1.$

Hence $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}.$$

From which $x_1 = 2 + \sqrt{3}$

and $x_2 = 2 - \sqrt{3}.$

Ex. 2. Solve $3x^2 + 16x - 12 = 0.$

Here $a = 3, b = 16, c = -12.$

Here $$x = \frac{-16 \pm \sqrt{(16)^2 - 4 \cdot 3(-12)}}{2 \cdot 3}.$$

From which $x_1 = \frac{2}{3}$, and $x_2 = -6.$

180. A quadratic equation may be proposed for solution which has no roots expressible in terms of the numbers of arithmetic or algebra so far as yet studied.

Example. Solve $x^2 + 4x = -8.$ (1)

By A, $x^2 + 4x + 4 = -4.$ (2)

Taking square roots, $x + 2 = \pm\sqrt{-4}.$ (3)

$\sqrt{-4}$ is unknown to us as a number symbol, since there is no number thus far considered whose square equals -4. (See Principle XI.) Such symbols are defined and used in the Advanced Course, and are called **imaginary numbers**. For us any quadratic equation which gives rise to such a solution is to be interpreted as stating some impossible condition.

EXERCISES

1. $7 - 3x = 5x^2.$
2. $51x - 33 = 3x^2.$
3. $14x + 8 - x^2 = 52 - 3x^2.$
4. $12 - 51x = 36 + 6x^2.$
5. $5x + x^2 + 8 = 0.$
6. $5x^2 - 31x = -6.$

7. $x^2 - 8 + 3x = -15x$.
8. $11x^2 - 49x + 57 = 0$.
9. $3x^2 + 18 - 16x = 5$.
10. $37 - 4x^2 - 12x = 79 - 5x^2$.
11. $10x^2 + 41 + 7x = 44$.
12. $45 + 3x^2 - 85 - 2x = 0$.

MISCELLANEOUS QUADRATICS

Solve as many as possible of the following equations by factoring. When this is not convenient, use the formula of § 179, or complete the square independently in each case. Show which equations state impossible conditions. Approximate all square roots to two decimal places.

1. $x^2 + 11x = 210$.
2. $5x^2 - 3x = 4$.
3. $7x + 3x^2 - 18 = 0$.
4. $2 = 5x + 7x^2$.
5. $6x - 11x^2 = -7$.
6. $-51 + 42x - 3x^2 = 0$.
7. $3x^2 + 3x = 2x + 4$.
8. $13 - 8x + 3x^2 = 0$.
9. $2x^2 + 11x = 32x - x^2 - 27$.
10. $176 + 3x - x^2 = 2x$.
11. $x^2 + 6x - 54 = 0$.
12. $5x^2 + 9x + 12 = 4x^2 + x$.
13. $2x^2 - 4x - 25 = 0$.
14. $7x^2 + 11x = 6$.
15. $2x^2 - 11x + 5 = 0$.
16. $2x^2 - 11x = 6$.
17. $25x - 95 = x^2$.
18. $11x^2 - 42x = 2$.
19. $x^2 - 8x - 4 = x - 22$.
20. $8x^2 + 5x = -8$.
21. $2x^2 + 3x - 3 = 12x + 2$.
22. $3x^2 - 7x = 10$.
23. $17x + 31 + 2x^2 = 0$.
24. $18 - 41x = 3 + x^2$.
25. $10x + 25 = 5 - 2x - x^2$.
26. $3x - 59 + x^2 = 0$.
27. $5x^2 + 7x - 6 = 0$.
28. $x^2 + 12 = 7x$.
29. $8x - 5x^2 = 2$.
30. $5x + 3x^2 - 22 = 0$.
31. $50 + 20x + x^2 = 5x$.
32. $x^2 + x + 4 = 0$.
33. $20x + 2x^2 + 42 = 33x + x^2$.
34. $17x - 3x^2 = -6$.
35. $8x + 5x^2 = -2$.
36. $10 + 15x + x^2 = 26x$.
37. $3x^2 - 2x - 7 = 0$.
38. $5x^2 - 9x - 18 = 0$.
39. $7x - 7x^2 + 24 = 0$.
40. $31 + 2x + x^2 = 0$.
41. $7x^2 + 7x - 5x^2 + 20 = x^2 - 2x + 2$.

181. The solution of two equations in two variables, one of which is linear and the other quadratic, can be reduced to the solution of a quadratic equation in one variable. (See § 149.)

EXAMPLE. Solve
$$\begin{cases} x + y = 3, & (1) \\ 3x^2 - y^2 = 14. & (2) \end{cases}$$

From (1),
$$y = 3 - x. \qquad (3)$$

Substituting in (2) and reducing,
$$2x^2 + 6x - 23 = 0. \qquad (4)$$

Substituting in formula § 179, $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 2(-23)}}{4}$. (5)

Hence
$$x_1 = 2.21 \text{ and } x_2 = -5.21.$$

Substituting these values of x in (1) we have as the approximate roots,
$$\left.\begin{matrix} x_1 = 2.21 \\ y_1 = 0.79 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_2 = -5.21 \\ y_2 = 8.21 \end{matrix}\right\}.$$

y_1 and y_2 are here used to designate the values of y which correspond to x_1 and x_2 respectively.

In this manner solve the following equations simultaneously, finding in each case two pairs of roots. In the case of roots which are neither integers nor exact fractions, find the approximate results to two places of decimals.

1. $\begin{cases} x - y = 1. \\ x^2 + y^2 = 13. \end{cases}$

2. $\begin{cases} x + y = 9. \\ x^2 + y^2 = 41. \end{cases}$

3. $\begin{cases} x + y = 13. \\ xy = 42. \end{cases}$

4. $\begin{cases} 3x - y = 5. \\ x^2 + y^2 = 25. \end{cases}$

5. $\begin{cases} x + 4y = 26. \\ x^2 - y^2 = 11. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{9} - \dfrac{y^2}{4} = 3. \\ x - y = 4. \end{cases}$

7. $\begin{cases} x - y = 1. \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = \frac{1}{2}. \end{cases}$

8. $\begin{cases} 2x + y = 5. \\ 3x^2 - 5y^2 = 7. \end{cases}$

9. $\begin{cases} x - y = 3. \\ x^2 - 3y^2 = 13. \end{cases}$

10. $\begin{cases} x-3y=1. \\ y^2+2x^2=33. \end{cases}$

11. $\begin{cases} 3x-4y=1. \\ x^2-y^2=24. \end{cases}$

12. $\begin{cases} x+y=4. \\ 2x^2-3xy+y^2=8. \end{cases}$

13. $\begin{cases} x-y=1. \\ 4x^2+2xy-y^2=19. \end{cases}$

14. $\begin{cases} 5x+y=12. \\ 2x^2-3xy+y^2=0. \end{cases}$

15. $\begin{cases} x-2y=3. \\ 2y^2-x^2=4. \end{cases}$

16. $\begin{cases} x+y=9. \\ x^2-2y^2=-7. \end{cases}$

17. $\begin{cases} y-2x=5. \\ y^2-3xy=16. \end{cases}$

18. $\begin{cases} 2y-3x=0. \\ y^2+x^2=52. \end{cases}$

19. $\begin{cases} y-2x=5. \\ x^2+y^2=40. \end{cases}$

20. $\begin{cases} x-4y=12. \\ 3x^2+2xy-6y=44. \end{cases}$

21. $\begin{cases} y-3x=7. \\ 2x^2-3xy=4. \end{cases}$

PROBLEMS

In each problem find the two roots of the quadratic equation and determine whether both are applicable to the problem:

1. The area of a window is 2016 square inches and the length of the frame is 180 inches. Find the dimensions of the window.

2. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk when measured on the side next the street is 560 yards. Find the dimensions of the block.

3. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow is 376 rods. Find the dimensions of the field.

4. A rectangular blackboard contains 38 square feet and the length of the molding is 27 feet. Find the dimensions of the board.

5. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

6. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

7. A city block is 400 by 480 feet when measured to the outer edge of the sidewalk. At 4 cents per square foot it costs $416.64 to lay a sidewalk around the block. Find the width of the walk.

8. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must he cut to make 12 acres?

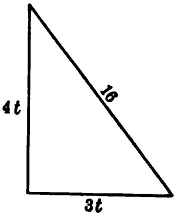
9. The sides of a right triangle are 6 and 8 inches respectively. How much must be added to each side so as to increase the hypotenuse 10 inches?

10. A rectangular lot is 16 by 12 rods. How wide a strip must be added to one end and one side to obtain a rectangular lot whose diagonal is 1 rod greater?

11. A picture is 15 inches by 20 inches. How wide a frame must be added to increase the diagonal 3 inches?

12. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) shall be increased by 100 feet. What will be the dimensions of the new field?

13. *A* starts north from a certain place going 4 miles per hour and *B* starts east from the same place at the same time going 3 miles per hour. In how many hours will they be 16 miles apart, the earth's surface being considered as a plane?



Let t equal the required number of hours.

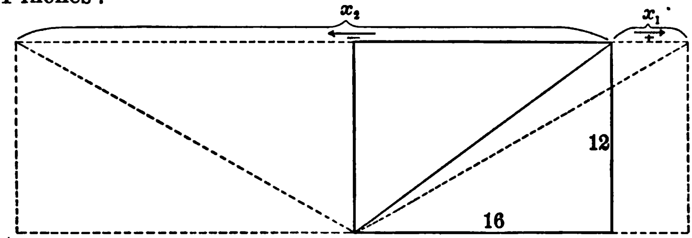
Then $(4t)^2 + (3t)^2 = 16^2 = 256.$

$$16t^2 + 9t^2 = 256.$$
$$25t^2 = 256.$$
$$5t = \pm 16.$$
$$t = \pm 3\tfrac{1}{5}.$$

The solution $t = -3\tfrac{1}{5}$ is not applicable to this problem.

14. In the preceding problem if A goes 5 miles per hour and B 4 miles per hour, in how many hours will they be 24 miles apart?

15. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal 4 inches?



Let x = number of inches to be added to the length. The diagonal of the original rectangle is $\sqrt{12^2 + 16^2} = 20$. Hence the diagonal of the required rectangle is 24.

Then $$12^2 + (16 + x)^2 = 24^2,$$
or $$x^2 + 32x - 176 = 0.$$
Solving, $$x_1 = -16 + 12\sqrt{3} = 4.78,$$
and $$x_2 = -16 - 12\sqrt{3} = -36.78.$$

The negative solution obtained here may be taken to mean that if the rectangle is extended in the *opposite* direction from the fixed corner, we shall get a rectangle which has the required diagonal. See the figure.

16. How much must the *width* of the rectangle in problem 15 be extended so as to increase the diagonal by 4?

17. A trunk 30 inches long is just large enough to permit an umbrella 36 inches long to lie diagonally on the bottom. How much must the length of the trunk be increased if it is to accommodate a gun 4 inches longer than the umbrella?

18. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.

19. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one-tenth of the whole piece?

20. A certain university campus is 100 rods long and 80 rods wide. There are two driveways running through the center of the campus at right angles to each other and parallel to the sides. What is the width of these driveways if their combined area is 356 square rods?

21. A farm is 320 rods long and 280 rods wide. There is a road 2 rods wide running around the boundary of the farm and lying entirely within it. There is also a road 2 rods wide running across the farm parallel to the ends. What is the area of the farm exclusive of the roads?

22. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park and a drive through the length of the park parallel to the sides. What is the width of the walk if the drive is three times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

23. The sum of the sides of a right triangle is 18 and the length of the hypotenuse is 16. Find the length of each side.

24. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Using 100 feet for the unit of measure the equations are

$$\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$$

25. The difference between the sides of a right triangle is 8 and the hypotenuse is 42. Find the lengths of the sides.

26. A room is 5 feet longer than it is wide and the distance between two opposite corners is 25 feet. Find the length and width of the room.

27. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of its hypotenuse and of the remaining side.

28. A vacant corner lot has a 50-foot frontage on one street. What is the frontage on the other street if the distance between opposite corners along the diagonal is 110 feet less than twice this frontage.

In an old Chinese arithmetic said to have been written about 2600 B.C., we find the following two problems in each of which the square of the unknown occurs but cancels.

29. In the middle of a square pond whose sides are 10 feet there grows a reed which reaches 1 foot above the water. When the reed is bent over to the side of the pond, it just reaches the top of the water. How deep is the water?

30. A bamboo reed 10 feet high is broken off so that the top reaches the ground just three feet from its base. How far from the ground is the reed broken off?

31. The sum of the squares of two consecutive integers is 13,945. Find the numbers.

32. The product of two consecutive integers is 4422. Find the numbers.

33. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.

34. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

35. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide.

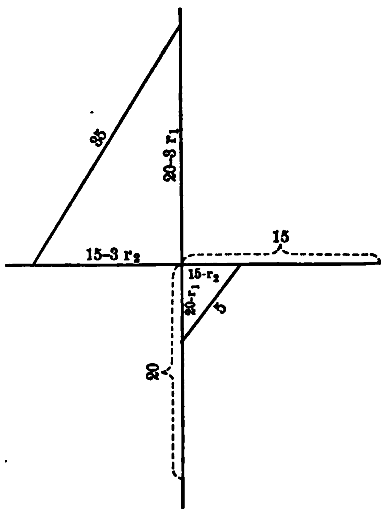
36. An open box whose bottom is a square has a lateral area which is 400 square inches more than the area of the bottom. Find the other dimensions of the box if it is 10 inches high. (By lateral area is meant the sum of the areas of the four sides.)

37. A box whose bottom is 4 times as long as it is wide has a lateral area 600 square inches less than 4 times the area of the bottom. Find the dimensions of the bottom if the box is 6 inches high.

38. A train approaching Chicago from the south at the rate of 50 miles per hour is 75 miles away when a train starts west from Chicago at the rate of 25 miles per hour. How long after the second train starts will they be 50 miles apart measured diagonally across the country?

If t is the number of hours required, then $(75-50\,t)^2+(25\,t)^2=(50)^2$. This may be written: $(25)^2\cdot(3-2\,t)^2+(25)^2\cdot t^2=2^2\cdot(25)^2$.

Hence dividing both members by $(25)^2$, we have $(3-2\,t)^2+t^2=4$.



39. An automobile running northward at the rate of 15 miles per hour is 20 miles south of the intersection with an east and west road. At the same time another automobile running westward on the cross-road at the rate of 20 miles per hour is 15 miles east of the crossing. How far apart (diagonally) will they be 15 minutes later? One hour later?

40. Under the conditions of problem 39 how long after the time speci-

fied will the automobiles be 10 miles apart? Is there more than one such position?

41. What are the rates of motion of the automobiles in problem 39 if one hour later they are 5 miles apart and 3 hours later they are 35 miles apart? (See the figure.)

If the rates of the automobiles are r_1 and r_2, then after 1 hour we have

$$(20 - r_1)^2 + (15 - r_2)^2 = 5^2, \qquad (1)$$

and after 3 hours we have $(20 - 3\,r_1)^2 + (15 - 3\,r_2)^2 = 35^2. \qquad (2)$

Simplifying (1) and (2), $r_1^2 + r_2^2 - 40\,r_1 - 30\,r_2 + 600 = 0, \qquad (3)$

and

$$9\,r_1^2 + 9\,r_2^2 - 120\,r_1 - 90\,r_2 - 600 = 0. \qquad (4)$$

Multiplying (3) by 9, subtracting from (4), and simplifying,

$$4\,r_1 + 3\,r_2 = 100. \qquad (5)$$

The solution of this problem may now be completed by solving (5) and (1) simultaneously.

NOTE. In equation (2), $20 - 3\,r_1$ and $15 - 3\,r_2$ are both negative numbers. This means that in this problem the distances north and west of the crossing are *negative*, while those south and east are *positive*.

REVIEW QUESTIONS

1. Define exponent. Explain the difference between an exponent and a coefficient.

2. Explain why and under what circumstances exponents are added in multiplication.

Show that $(a^2)^3 = a^6$, $(a^3)^4 = a^{12}$.

3. Explain the method of multiplying two monomials.

How is Principle III involved in Principle XV?

4. What is meant by factoring? Is the following expression factored? $x(a + b) + y(a + b)$. Why?

5. What are the characteristics of a trinomial square?

Are the following trinomials squares? If not, change one term in each so as to make it a square. $x^2 + xy + y^2$; $x^4 + x^2y^2 + y^4$; $a^2 - 2\,ab - b^2$; $4\,a^2 + 4\,ab + 4\,b^2$.

6. What are the factors of the difference of two squares? Factor $x^6 - y^6$ as the difference of two squares.

7. What are factors of the difference of two cubes? Factor $x^6 - y^6$ as the difference of two cubes.

8. What are the factors of the sum of two cubes? Factor $x^6 + y^6$ as the sum of two cubes.

9. Explain how to factor a trinomial by inspecting the end products and cross-products of two binomials.

By this method factor,

$$3x^2 - 7x - 10;\ x^2 - 9x + 18;\ 3x^2 + 5x - 12.$$

10. By means of the following examples explain the process of factoring by grouping.

$$x^2 + ax + bx + ab;\ x^3 - x - 3x^2 + 3.$$

11. How may a quadratic equation be solved by factoring? Why is it essential that one member of the equation be zero when the other member is factored?

12. Explain the method of solving the quadratic equation by completing the square.

13. How many roots has a quadratic equation? When does the solution of a quadratic equation indicate that the conditions of a problem are impossible?

14. Explain why and under what circumstances exponents are subtracted in division.

15. Explain the method of finding the quotient of two monomials. Show how Principle V is involved in Principle XVII.

16. State Principle XVIII. Given $\sqrt{7} = 2.646$, find $\sqrt{28}$ by means of this principle.

17. Explain how a number in Arabic figures is divided into groups for the purpose of finding its square root.

18. Show how the value of the following may be approximated by finding only one square root.

$$5\sqrt{20} + 2\sqrt{45} - 3\sqrt{80} + 2\sqrt{\tfrac{1}{5}}.$$

CHAPTER VIII

LITERAL FRACTIONS

COMMON FACTORS

182. If a number is a factor of each of two or more numbers, it is said to be a **common factor** of these numbers.

Thus, 8 is a common factor of 16 and 48, and 12 is a common factor of 12, 36, and 48.

If each of a given set of numbers is factored into prime factors, any common factor which they may have is at once apparent.

Illustrative Example. Find the common factors of 72, 96, 240, and 288.

Factoring, we have
$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2.$$
$$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^5 \cdot 3.$$
$$240 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^4 \cdot 3 \cdot 5.$$
$$288 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^5 \cdot 3^2.$$

The common **prime** factors are 2, 2, 2, and 3. The other common factors are the various combinations of these, namely, $2 \cdot 2 = 4$, $2 \cdot 2 \cdot 2 = 8$, $2 \cdot 3 = 6$, $4 \cdot 3 = 12$, and $8 \cdot 3 = 24$. 24 is, therefore, the **greatest common factor** of these numbers.

To find common factors of literal expressions we proceed in the same manner.

Illustrative Example. Find the common factors of $x + y$, $x^2 - y^2$, and $x^2 + 2xy + y^2$.

Factoring, we have
$$x + y = x + y.$$
$$x^2 - y^2 = (x + y)(x - y).$$
$$x^2 + 2xy + y^2 = (x + y)(x + y).$$

Hence, $x + y$ is the only common factor of these expressions.

Illustrative Example. Find the common factors of

$$10x^2+20xy+10y^2,\ 5(x+y)(x^2-y^2),\ \text{and}\ 15(x+y)(x^3+y^3).$$

Factoring, $10x^2+20xy+10y^2=2\cdot 5(x+y)(x+y).$

$$5(x+y)(x^2-y^2)=5(x+y)(x+y)(x-y).$$

$$15(x+y)(x^3+y^3)=3\cdot 5\,(x+y)(x+y)(x^2-xy+y^2).$$

The common prime factors are 5, $x+y$, and $x+y$. The other factors, obtained by combining these, are $5(x+y)$, $(x+y)^2$, and $5(x+y)^2$. The last factor, $5(x^2+2xy+y^2)$, is called the highest common factor.

The name *highest* instead of *greatest* is used in algebra referring to the degree of the factor. Thus x^2 is of higher degree than x, although if $x=\frac{1}{2}$, x^2 is not greater than x.

183. Definition. In general that common factor which contains the greatest number of prime factors is the **highest common factor.** This is usually abbreviated to H. C. F. (See § 130.)

EXERCISES

Find the H. C. F. of the following sets of expressions:

1. $x-y,\ x^2-y^2,\ x^2-2xy+y^2.$
2. $x^2+2x+1,\ 3x+6x^2+3x^3.$
3. $x^2+4x+4,\ x^2-6x-16.$
4. $x^2-8x+16,\ x^2+10x-56.$
5. $a^3-b^3,\ a^2-2ab+b^2.$
6. $x^3+y^3,\ x^2-y^2,\ x^2+2xy+y^2.$
7. $x^2-7x+12,\ ax-3a-bx+3b.$
8. $a^2-13a+42,\ a^3-216,\ a^2-a-30.$
9. $27+y^3,\ y^2+9y+18,\ y^2-9.$
10. $b^2+7b-30,\ b^2+11b-42,\ b^2-b-6.$

11. $a^3 + 2a^2 + a,\ a^2 + a,\ a^3 + 5a^2 + 4a.$
12. $x^3 + y^3,\ x^3 + x^2y + xy^2 + y^3.$
13. $x^4 + 3x^3 + 2x^2,\ x^3 + x^2,\ x^4 + 7x^3 + 6x^2.$
14. $x^2 - 11x + 30,\ xz - 5z + x^2 - 5x.$
15. $m^3 - n^3,\ 2x^2m^2 + 2x^2mn + 2x^2n^2.$
16. $x^2 - 1,\ x^3 - 1,\ x^2 - 13x + 12.$
17. $1 - 64x^3,\ 1 - 16x^2,\ 5 - 2z - 20x + 8xz.$
18. $1 + 125a^3,\ 1 + 10a + 25a^2,\ 1 - 25a^2.$
19. $ac - ax + 3bc - 3bx,\ a^3 + 27b^3.$
20. $5c - 2,\ 5ac + 20c - 2a - 8.$
21. $4x^4 - x^2,\ 2x^4 + x^3 - x^2,\ 2x^4 - 3x^3 + x^2.$
22. $3a^3 - 3a,\ 3a^3 - 6a^2 + 3a,\ 6a^3 + 12a^2 - 15a.$
23. $6x - 10xy + 4xy^2,\ 18x - 8xy^2,\ 54x - 16xy^3.$
24. $3x^5 + 9x^4 - 3x^3,\ 5x^2y^2 + 15xy^2 - 5y^2,\ 7ax^2 + 21ax - 7a.$
25. $18x^3 - 57x^2 + 30x,\ 9x^3 - 15x^2 + 6x,\ 18x^3 - 39x^2 + 18x.$

COMMON MULTIPLES

184. A number is said to be a **multiple of any of its factors.** In particular any number is a multiple of itself and of one.

Thus, 18 is a multiple of 1, 2, 3, 6, 9, and 18, but not of 12. $3a^2x^2$ is a multiple of 3, $3x$, $3x^2$, etc.

Since a multiple of a number is divisible by that number, it must contain as a factor every factor of that number.

E.g. 108 is a multiple of 54 and contains as factors all the factors of 54, namely 3, 3, 3, and 2, and also 2, 6, 9, 18, and 54.

Definition. A number is a **common multiple** of two or more numbers if it is a multiple of each of the numbers.

Thus, 18 is a common multiple of 6, 9 and 18. Evidently $3 \cdot 18$, $4 \cdot 18$, $5 \cdot 18$, etc. are also common multiples of 6, 9 and 18. Of all these common multiples 18 is the *smallest* and is called the *least* or *lowest* common multiple.

185. The process of finding the least common multiple of a set of numbers in Arabic figures is shown as follows:

Illustrative Example. Find the least common multiple of 16, 24, and 98.

Finding the prime factors of each number,

$$16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$
$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$$
$$98 = 2 \cdot 7 \cdot 7 \quad = 2 \cdot 7^2$$

Any multiple of 16 contains all its prime factors, namely, 2, 2, 2, and 2. Any common multiple of 16 and 24 contains in addition to the prime factors of 16 any prime factors of 24 not in 16, namely 3. Any common multiple of 16, 24, and 98 contains in addition to 2, 2, 2, 2 and 3 those prime factors of 98 not in 16 or 24, namely 7 and 7. Hence $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 \cdot 7 = 1176$ is a *common* multiple of 16, 24, and 98. Moreover, it is the *least common multiple* because no unnecessary factor has been included.

We proceed with a set of literal expressions in the same manner as above.

Illustrative Example. Find the L. C. M. of

$$x^2 - y^2;\ x^2 + 2xy + y^2;\ x^2 - 2xy + y^2.$$

Factoring,
$$x^2 - y^2 = (x - y)(x + y). \tag{1}$$
$$x^2 + 2xy + y^2 = (x + y)(x + y). \tag{2}$$
$$x^2 - 2xy + y^2 = (x - y)(x - y). \tag{3}$$

In order that an expression may be a multiple of (1) it must contain the factors $x - y$ and $x + y$. To be a common multiple of (1) and (2) it must contain a second factor $x + y$, giving $(x - y)$, $(x + y)$, $(x + y)$. To be a common multiple of (1), (2) and (3) it must contain a second factor $x - y$, giving $(x - y), (x + y)$, $(x + y), (x - y)$. The product thus found contains the fewest prime factors possible in order to be a common multiple of (1), (2), and (3). Hence $(x - y)(x + y)(x + y)(x - y) = (x - y)^2 (x + y)^2$ is called the lowest common multiple of (1), (2), and (3), since it is the common multiple of lowest degree.

In general, the process may be described as follows: to obtain the lowest common multiple of a set of expressions, factor each expression into prime factors; use all factors of the first expression together with those factors of the second which are not in the first, those of the third which are not in the first and second, etc. It is evident that in this manner we obtain a product which is a common multiple of the given expressions, but such that if any one of these factors is omitted, it will cease to be a multiple of some one of the expressions; that is, it will no longer be a common multiple of them *all*.

Thus, if in the example above either of the factors $x-y$ is omitted, the product will no longer be a multiple of $x^2-2xy+y^2$.

186. Definition. We now define the **lowest common multiple** of a set of expressions as that common multiple which contains the smallest number of prime factors. The lowest common multiple is usually abbreviated to L. C. M.

EXERCISES

Find the L. C. M. of the following expressions:

1. $2\cdot3\cdot4;\ 3\cdot7\cdot8;\ 2^3\cdot3\cdot4$.
2. $5x^2y^4,\ 10x^3y,\ 25x^2y$.
3. $2ab,\ 6a^2,\ 4b^2c$.
4. $x^2-y^2,\ x^2-2xy+y^2$.
5. $x-y,\ x+y,\ x^2-y^2$.
6. $4-x^2,\ 2-x,\ 2+x$.
7. $a^2+2ab+b^2,\ a^2-2ab+b^2$.
8. $x^2+3x+2,\ x^2-4,\ x^2-1$.
9. $25x^2-1,\ 125x^3-1$.
10. $2x^2-7x+6,\ 4x^2-11x+6$.
11. $x^3-y^3,\ x-y,\ x^2+xy+y^2$.
12. $x^3-y^3,\ x^3+y^3,\ x^2-y^2$.
13. $5x^2+7x-6,\ x^2-15x-34$.
14. $x^3+y^3,\ x^2-y^2,\ (x-y)^2$.
15. $3abc,\ a^2-4ac+4c^2,\ a-2c$.
16. $x^2-1,\ x+1,\ x^2+8x+7$.
17. $4x^3y-44x^2y+120xy,\ 3a^3x^2-22a^3x+35a^3$.
18. $x^2+2xy+y^2,\ 2ax^2-10ax+12a$.

19. $3bx^2 - 21bx + 36b,\ x^2 - 5x + 4.$

20. $5a^2b^2 - 5a^2c^2,\ b^2 + 2bc + b + c + c^2.$

21. $15c^2ax^2 + 16c^2ax + c^2a,\ 2cax^2 + 10cax + 8ca.$

22. $ac - 4a + bc - 4b,\ \ ax + ac + bx + bc.$

23. $x^2 + 5x + 4,\ x^2 + 7x + 12,\ x^2 + 5x + 6.$

24. $x^2 + 2x + 1,\ x^2 - 2x + 1,\ x^2 - 1.$

REDUCTION OF FRACTIONS TO LOWEST TERMS

187. In arithmetic the denominator of a fraction is usually regarded as indicating the number of equal parts into which a unit is divided, while the numerator designates a certain number of these parts.

Thus, $\frac{2}{3}$ means 2 of the 3 equal parts of a unit.

However, a fraction such as $\dfrac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\dfrac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$. *I.e.* $\dfrac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

In algebra any fraction is usually regarded as an **indicated division** in which the numerator is the dividend and the denominator is the divisor.

Thus, $\dfrac{a}{b}$ is understood to mean $a \div b$.

The numerator and denominator are together called **the terms of the fraction.**

In case the numerator and denominator have common factors, these may be removed, without changing the value of the fraction, by means of Principle XVII, as applied in § 157.

Thus, $\dfrac{2 \cdot 3 \cdot 4 \cdot 5}{3 \cdot 7 \cdot 11} = \dfrac{2 \cdot 4 \cdot 5}{7 \cdot 11}$, where the common factor 3 is canceled.

Similarly $\frac{2^3 \cdot 3^2 \cdot 4x}{2^4 \cdot 3 \cdot 4^2} = \frac{3x}{2 \cdot 4}$, the factors 2^3, 3, and 4 being canceled; and $\frac{(x^2 - 7x + 12)}{(x^2 - 5x + 6)} = \frac{(x-3)(x-4)}{(x-2)(x-3)} = \frac{(x-4)}{(x-2)}$.

If the terms of a fraction have no common factor, the fraction is said to be in its **lowest terms.** Evidently the process of canceling common factors in the numerator and denominator may always be continued until the fraction is reduced to its lowest terms.

EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{3 \cdot 9^2 \cdot 2^6}{2^4 \cdot 5^3 \cdot 9^4}$.
2. $\frac{4a^4b^{12}c^3}{8a^3b^4c^4}$.
3. $\frac{x^2y^3z^4}{xy^2z^3}$.
4. $\frac{a^4b^3}{a^2b^2}$.
5. $\frac{x^2 + 2xy + y^2}{x^2 - y^2}$.
6. $\frac{x^2 + 7x - 30}{x^2 - 7x + 12}$.
7. $\frac{x^3 - y^3}{2x^2 - 3xy + y^2}$.
8. $\frac{64 - b^3}{16 - 8b + b^2}$.
9. $\frac{x^3 + 27z^3}{xy - 5x + 3yz - 15z}$.
10. $\frac{1 - 216c^3}{x - 4y - 6cx + 24cy}$.
11. $\frac{14bz - 2bx + ax - 7az}{x^2 - 49z^2}$.
12. $\frac{3a^2 - 29a + 56}{63 - 9a - 7m + ma}$.
13. $\frac{a(x-y)^3}{(x^2 - y^2)(x - y)}$.
14. $\frac{x^3 + 27}{4x^2 + 24x + 36}$.
15. $\frac{a^2 - 3a - 3b + ab}{(a^2 - b^2)(a - 3)}$.
16. $\frac{2^7 \cdot 3^5 \cdot 5^4 - 2^6 \cdot 3^4 \cdot 5^7}{2^4 \cdot 3^2 \cdot 5^5 - 2^5 \cdot 3^2 \cdot 5^2}$.
17. $\frac{5x^3y^4 - 12x^2y^5 + 7x^3y^2}{6x^5y^2 + 3x^2y^2}$.
18. $\frac{5c + 10b - bc - 2b^2}{8c^3 + 64b^3}$.
19. $\frac{4x^4 - 28x^3 + 48x^2}{2x^4 - 8x^3 + 6x^2}$.
20. $\frac{9a^2b^4 + 18a^2b^3c + 9a^2b^2c^2}{3ab^3 - 3abc^2}$.
21. $\frac{7xy^2 - 133xy + 126x}{15xy^2 - 36xy + 21x}$.

22. $\dfrac{20x^3+20x^2y+5xy^2}{60x^5-15x^3y^2}$.

23. $\dfrac{3ab^4-3ab^2c^2}{27a^3b^2+27a^3bc}$.

24. $\dfrac{4a^3-42a^2+20a}{2a^4b^6-20a^3b^6}$.

25. $\dfrac{(x-1)(x-2)(x-3)(x-4)}{(x-1)(x-3)(x-3)(x-4)}$.

26. $\dfrac{(x^2-y^2)(x^2+2xy+y^2)}{(x^2-2xy+y^2)(x+y)}$.

27. $\dfrac{(x^2-1)(x^2+1)(3x^2+3)}{3(x^4-1)}$.

28. $\dfrac{(3a^4-3a^2b^2)(a^2+13a+42)}{(a^2+3a+2)(a^2+5a-6)}$.

29. $\dfrac{2^3\cdot 4^3\cdot 5^4(x^2-b^2)(x^2+19x+90)}{2^2\cdot 4\cdot 5^3(x^2+9x-10)(x^2+10x+9)}$.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

188. Since we have just seen that a common factor may be *removed* from the numerator and denominator of a fraction without changing the value of the fraction, it is evident that a factor may be *introduced* into both terms without changing the value of the fraction.

Thus since $\dfrac{3\cdot 4}{3\cdot 5}$ is reduced to $\dfrac{4}{5}$ by removing the factor 3 from both numerator and denominator, so $\dfrac{4}{5}$ is changed to $\dfrac{3\cdot 4}{3\cdot 5}$ by introducing the factor 3 in both the terms. Likewise any other factor may be introduced. *E.g.* $\dfrac{4}{5}=\dfrac{4\cdot 7}{5\cdot 7}$, $\dfrac{a-b}{a+b}=\dfrac{(a-b)(a+b)}{(a+b)(a+b)}=\dfrac{a^2-b^2}{(a+b)^2}$.

In this manner any fraction may be changed into an equal fraction whose denominator is any given multiple of the denominator of the given fraction.

Thus, $\frac{3}{4}$ can be changed into a fraction whose denominator is 72 (a multiple of 4) by multiplying both terms by 18, *i.e.* $\dfrac{3}{4}=\dfrac{3\cdot 18}{4\cdot 18}$;

and $\dfrac{3a-2b}{x-y}$ can be changed into a fraction whose denominator is x^2-y^2 by multiplying both of its terms by $x+y$, *i.e.*,

$$\frac{3a-2b}{x-y}=\frac{(3a-2b)(x+y)}{x^2-y^2}.$$

Any two or more fractions may therefore be changed into respectively equal fractions which shall have a *common denominator*, namely, a common multiple of the denominators of the given fractions.

Illustrative Example. Reduce $\dfrac{x-1}{x+1}$, $\dfrac{x+1}{x-1}$, $\dfrac{2x+3}{x^2-1}$ to fractions having a common denominator.

The L. C. M. of the denominators is $(x-1)(x+1)$. Multiply the numerator and denominator of each fraction by an expression which will make the denominator of each new fraction $(x-1)(x+1)$.

Thus,
$$\frac{x-1}{x+1}=\frac{(x-1)(x-1)}{(x+1)(x-1)}=\frac{x^2-2x+1}{(x+1)(x-1)};$$
$$\frac{x+1}{x-1}=\frac{(x+1)(x+1)}{(x-1)(x+1)}=\frac{(x+1)^2}{(x+1)(x-1)};$$
$$\frac{2x+3}{x^2-1}=\frac{2x+3}{(x+1)(x-1)}.$$

It is best to *indicate* the multiplication in the common denominator, since this makes it more easily apparent by what expression the numerator and denominator of a fraction must be multiplied in order to reduce it to a fraction with the required denominator.

It should be noticed that there are three signs in connection with a fraction, the sign of the fraction itself, the sign of the numerator, and the sign of the denominator. It follows from the law of signs in division (Principle XII) that any two of these signs may be changed simultaneously without changing the value of the fraction.

E.g.
$$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}.$$

This is useful in cases like the following:

Reduce $\frac{x+1}{1-x}, \frac{x}{x^2-1}$, and $\frac{1}{x+1}$ to fractions having a common denominator.

$$\frac{x+1}{1-x}=\frac{-x-1}{x-1}=\frac{(x+1)(-x-1)}{(x+1)(x-1)}=\frac{-x^2-2x-1}{x^2-1},$$

$$\frac{x}{x^2-1}=\frac{x}{x^2-1}, \text{ and } \frac{1}{x+1}=\frac{x-1}{(x+1)(x-1)}=\frac{x-1}{x^2-1}.$$

EXERCISES

Reduce each of the following sets of fractions to equivalent fractions having a common denominator.

1. $\frac{x+3}{x-y}, \frac{4}{x^2-2xy+y^2}.$

2. $\frac{a-1}{a^2-b^2}, \frac{a+1}{a^2+2ab+b^2}.$

3. $\frac{3-x}{x^2-9x+20}, \frac{x+4}{7x^2-26x-8}.$

4. $\frac{2}{3-x}, \frac{x-1}{x+1}, \frac{x+1}{x-3}.$

5. $\frac{a+b}{b-a}, \frac{a-b}{(a+b)^2}, \frac{a}{a^2-b^2}.$

6. $\frac{1}{x^2-3x-4}, \frac{1}{x^2+3x+2}.$

7. $\frac{a+1}{a^2-2ab+b^2}, \frac{a-1}{a^2+2ab+b^2}.$

8. $\frac{1}{a^3-b^3}, \frac{1}{b-a}, \frac{1}{a^2+ab+b^2}.$

9. $\frac{a}{5a^2-4a-12}, \frac{b}{a^2+4a-12}, \frac{c}{a-2}.$

10. $\frac{x-2}{x^2-5x-6}, \frac{x+2}{x^2+12x-108}, \frac{x-1}{x^2+19x+18}.$

In each of the following exercises reduce the given fractions to a common denominator and check by substituting a convenient number for each letter, taking care that no denominator becomes zero:

11. $\frac{mr}{m-1}, \frac{d}{1+n}$.

12. $\frac{Rr}{(R+r)(m-1)}, \frac{1}{R+r}$.

13. $\frac{CS}{R+RS}, \frac{Rr+Sr+SR}{RSs}$.

14. $\frac{a}{n-a}, \frac{b}{n-b}$.

15. $\frac{W(T-Q)}{w(Q-t)}, \frac{V}{w}$.

16. $\frac{x(T-t)}{W(Q-T)}, \frac{W(T-Q)}{w(Q-t)}$.

17. $\frac{P}{W(p-2t+1)}, \frac{d(V-w)}{W-w}$.

18. $\frac{V}{V-v}, \frac{V}{V+v}, \frac{1}{V^2-v^2}$.

19. $\frac{RA}{a-A}, \frac{1}{R+r}, \frac{1}{A-a}$.

20. $\frac{R}{x}, \frac{r}{y}, \frac{1}{x-y}, \frac{Rx}{x+y}$.

189. Since any number may be written as a fraction with the denominator 1, the above process may be used to reduce an integral expression to the form of a fraction having any desired denominator.

Thus, $3=\frac{3\cdot 5}{5}$; $x-y=\frac{(x-y)(x^2-1)}{x^2-1}$, etc.

It is sometimes convenient to reduce expressions, some of which are not fractions, to the form of fractions having a common denominator.

Illustrative Example. Reduce $5x$, $\frac{5x-1}{x^2-1}$, $\frac{2x-y}{x-1}$, to fractions having a common denominator. The lowest common denominator is x^2-1.

Thus, $5x=\frac{5x(x^2-1)}{x^2-1}=\frac{5x^3-5x}{x^2-1}$,

$$\frac{5x-1}{x^2-1}=\frac{5x-1}{x^2-1},$$

$$\frac{2x-y}{x-1}=\frac{(2x-y)(x+1)}{(x-1)(x+1)}=\frac{2x^2+2x-yx-y}{x^2-1}.$$

EXERCISES

Reduce the following mixed expressions to fractions having a common denominator:

1. $5,\ x-y,\ \dfrac{5x-3}{x^2+2xy+y^2}$.

2. $\dfrac{3a-c}{x-y},\ \dfrac{2b-c}{x+y},\ 2c+2$.

3. $1+a+a^2,\ \dfrac{a+1}{a-1}$.

4. $x^2+xy+y^2,\ \dfrac{x+y}{x-y}$.

5. $x^2-xy+y^2,\ x^2-y^2,\ \dfrac{1}{x+y}$.

6. $x^4+x^2y^2+y^4,\ \dfrac{x}{x-y},\ \dfrac{y}{x+y}$.

7. $3a-2b-c,\ \dfrac{5}{a-b},\ \dfrac{2}{b-c}$.

8. $x^4-1,\ x^2-1,\ \dfrac{x+1}{x-1}$.

9. $x^2+2xy+y^2,\ \dfrac{1}{x+y},\ \dfrac{1}{1-x}$.

10. $x+y,\ x-y,\ \dfrac{x-y}{x^2+y^2},\ \dfrac{x+1}{x-y}$.

In each of the following reduce the expressions to the form of fractions having a common denominator and check the results by substituting convenient numbers for the letters:

11. $\dfrac{RA}{a-A},\ r$.

12. $\dfrac{RA}{a-A},\ R-r$.

13. $T,\ \dfrac{Q+t}{2}$.

14. $\dfrac{W(q-t)}{w},\ sT,\ \dfrac{s(t-q)}{2}$.

15. $H,\ \dfrac{hd}{D}$.

ADDITION AND SUBTRACTION OF FRACTIONS

190. Fractions having a common denominator may be added or subtracted, exactly as in arithmetic, by adding or subtracting the numerators and dividing the result by the common denominator.

For we have, by Principle VI, $\frac{16+20}{4}=\frac{16}{4}+\frac{20}{4}=4+5$. Likewise $\frac{3+5}{4}=\frac{3}{4}+\frac{5}{4}$, the division being indicated in this case. Hence, reading this identity from right to left, we have $\frac{3}{4}+\frac{5}{4}=\frac{3+5}{4}$.

Likewise $\frac{6}{x-y}-\frac{4}{x-y}=\frac{6-4}{x-y}=\frac{2}{x-y}$. In general,

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}, \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

If fractions which are to be added or subtracted do not have a common denominator, they must be reduced to this form

EXAMPLE. Add $\frac{a-b}{a+b}$ and $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

Reducing the fractions to the common denominator

$$(a-b)(a-b)(a+b),$$

we have

$$\frac{a-b}{a+b}=\frac{(a-b)(a-b)(a-b)}{(a+b)(a-b)(a-b)}=\frac{a^3-3a^2b+3ab^2-b^3}{(a+b)(a-b)(a-b)},$$

and $$\frac{a^2+2ab+b^2}{a^2-2ab+b^2}=\frac{(a+b)(a^2+2ab+b^2)}{(a+b)(a-b)(a-b)}=\frac{a^3+3a^2b+3ab^2+b^3}{(a+b)(a-b)(a-b)}.$$

Adding the numerators, we have $2a^3+6ab^2$; whence the sum of the fractions is

$$\frac{2a^3+6ab^2}{(a+b)(a-b)(a-b)}.$$

EXERCISES

Perform the following additions and subtractions:

1. $\frac{3}{4}+\frac{2}{7}+\frac{a+b}{3a-b}$.

2. $\frac{x-y}{(x+y)^2}-\frac{x-y}{x^2-y^2}$.

3. $\frac{x^2-9x+18}{x^2-13x+36}+\frac{x}{4-x}$.

4. $\frac{2}{3}+\frac{a}{a+b}+\frac{b}{a-b}$.

5. $\frac{3}{2^3\cdot 3^2}+\frac{5}{2^2\cdot 3^4}-\frac{2}{2^4\cdot 3^3}$.

6. $\frac{a^2-9b^2}{a^2+6b+9b^2}-\frac{a^2-6ab}{a^2-9b^2}$.

7. $\frac{x+y}{x-y}+\frac{x-y}{x+y}-2$.

8. $\frac{2x^2}{x^2-y^2}-\frac{y}{x-y}-\frac{y}{x+y}$.

9. $\frac{a+1}{a^2+a+1}-\frac{a-1}{a^2-a+1}$.

10. $\frac{b}{1-b}+\frac{b}{1+b^2}-\frac{b^2}{1-b^2}$.

11. $\frac{x+1}{x-2}+\frac{x+1}{x+2}+\frac{3x+2}{x^2-4}$.

12. $\frac{x-1}{x+1}-\frac{x+1}{x-1}+\frac{x^2-5}{x^2-1}$.

13. $\frac{y^2}{y^2-1}+\frac{y}{y+1}-\frac{y}{1-y}$.

14. $\frac{1}{x}-\frac{1}{y}-\frac{1}{x-y}+\frac{-1}{x+y}$.

15. $\frac{1}{2}-\frac{b}{b-a}+\frac{2a^2}{b^2-a^2}+\frac{-a}{b+a}$.

16. $\frac{x^2+4xy}{x^3+y^3}+\frac{1}{x+y}-\frac{x}{x^2-xy+y^2}$.

17. $\frac{1}{1-x^3}+\frac{1}{1-x}-\frac{1}{1+x+x^2}$.

18. $\frac{a-3}{a^2-3a+2}-\frac{a-1}{a^2-5a+6}+\frac{a}{a^2-4a+3}$.

19. $\dfrac{x}{x^2-5x-14}+\dfrac{2}{x-7}-\dfrac{x}{x^2-9x+14}.$

20. $\dfrac{a}{ac+ad-bc-bd}-\dfrac{b}{a^2-2ab+b^2}.$

21. $\dfrac{2}{a-5}+\dfrac{39}{a^2+3a-40}+\dfrac{3}{a+8}.$

22. $\dfrac{1}{y^2+8y+16}-\dfrac{1}{y(y+4)}+\dfrac{4}{y^2(y+4)}.$

23. $\dfrac{x}{x^2+4x-60}+\dfrac{x}{x^2-4x-12}.$

24. $\dfrac{a-c}{a^2-c^2}+\dfrac{c-a}{a^2+2ac+c^2}-\dfrac{2}{a-c}.$

25. $\dfrac{9}{x^2+7x-18}-\dfrac{8}{x^2+6x-16}.$

26. $\dfrac{a+2}{a^2-a-6}+\dfrac{a-4}{a^2-7a+12}-\dfrac{a+2}{a^2-2a-8}.$

27. $\dfrac{2}{x^2-11x+30}-\dfrac{1}{x^2-36}+\dfrac{1}{x^2-25}.$

28. $\dfrac{1}{(x-1)(x+2)}+\dfrac{1}{(x+2)(x-3)}+\dfrac{1}{(x-1)(3-x)}.$

29. $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-a)(c-d)}+\dfrac{1}{(b-c)(c-d)}.$

30. $\dfrac{4}{a-3}-\dfrac{a-1}{a^2+3a+9}+\dfrac{a^2-38a-3}{a^3-27}.$

MULTIPLICATION AND DIVISION OF FRACTIONS

191. The product of a fraction and an integer. In arithmetic the product of a fraction and an integer is obtained by multiplying the numerator of the fraction by the integer.

Thus, $\frac{2}{3} \times 4 = \frac{2 \cdot 4}{3}$ and $7 \times \frac{4}{5} = \frac{7 \cdot 4}{5}$.

Since in algebra a fraction is an indicated quotient, and since multiplying the dividend multiplies the quotient, it follows that in algebra also the product of a fraction and an integral expression is obtained by multiplying the numerator by the integral expression.

That is, in general, $a \cdot \frac{b}{c} = \frac{b \cdot a}{c} = \frac{ab}{c}$.

It is best to factor completely the expressions to be multiplied and keep them in the factored form until all possible cancellations have been made.

EXERCISES

Find the following indicated products and reduce the fractions to the simplest form.

1. $(1-a) \times \frac{1+a+a^2}{a-1}$.
2. $(x^3-y^3) \times \frac{x+y}{x-y}$.
3. $(x^2-2xa+a^2) \times \frac{x+a}{x-a}$.
4. $\frac{3x-1}{x^2-5x+6} \times (x^2-11x+18)$.
5. $\frac{a^2-4a-3}{a^2-8a+16} \times (a^2-5a+4)$.
6. $(x^2+9x+18) \times \frac{x-5}{x^2-2x-15}$.
7. $(1-x^3) \times \frac{1-x}{1+x+x^2}$.
8. $(27a^3-1) \times \frac{a+1}{9a^2+3a+1}$.
9. $(a^2+ab+b^2) \times \frac{a-b}{a^3-b^3}$.
10. $(1-a+a^2) \times \frac{a+1}{a^3+1}$.

192. In arithmetic a fraction is divided by an integer by multiplying its denominator or dividing its numerator by the integer.

Thus, $\frac{1}{3} \div 7 = \frac{1}{3 \cdot 7}$; $\frac{14}{5} \div 7 = \frac{14 \div 7}{5} = \frac{2}{5}$.

Since multiplying the divisor or dividing the dividend divides the quotient, it follows that an algebraic fraction is divided by an integral expression by dividing the numerator or multiplying the denominator by the integral expression.

That is, in general, $\frac{a}{b} \div c = \frac{a}{bc} = \frac{a \div c}{b}$.

EXERCISES

Find the following indicated quotients and reduce the fractions to their lowest terms:

1. $\frac{x^3 - y^3}{x + y} \div (x^2 + xy + y^2)$.

2. $\frac{x^3 + y^3}{x - y} \div (x^2 - y^2)$.

3. $\frac{x^2 + 4x + 4}{x^2 - 2x + 1} \div (x^2 - 4)$.

4. $\frac{1 - 27x^3}{1 - 9x^2} \div (1 + 3x + 9x^2)$.

5. $\frac{x^2 + 2x - 35}{x^2 + 10x + 21} \div (x^2 - 4x - 5)$.

6. $\frac{x^2 - 16x + 39}{x^2 - 8x + 15} \div (x^2 - x - 156)$.

7. $\frac{ac + bc - ad - bd}{xy - 4x - 3y + 12} \div (cx - 3c - dx + 3d)$.

8. $\frac{x^2 + ax + bx + ab}{x^2 + ax - 3x - 3b} \div (x^2 + ax - 5x - 5a)$.

9. $\frac{mr + ms - nr - ns}{mx - m - nx + n} \div (3r - xr + 3s - xs)$.

10. $\frac{x^2 - 3x - 88}{x^2 - 9x - 22} \div (x^2 + 9x + 2)$.

TO MULTIPLY A FRACTION BY A FRACTION

193. In arithmetic, to multiply a number by the quotient of two numbers is the same as to multiply by the dividend and then divide the product by the divisor.

E.g. $10 \cdot \frac{18}{3} = 10 \cdot 6 = 60$; or, $10 \cdot \frac{18}{3} = (10 \cdot 18) \div 3 = 60.$

Likewise, $\frac{2}{3} \cdot \frac{5}{7} = \left(\frac{2}{3} \cdot 5\right) \div 7 = \frac{2 \cdot 5}{3} \div 7 = \frac{2 \cdot 5}{3 \cdot 7}.$

Since in algebra a fraction is an indicated quotient, to multiply a number by an algebraic fraction we multiply by the numerator and divide the product by the denominator.

Thus, $$\frac{a}{b} \cdot \frac{c}{d} = \left(\frac{a}{b} \cdot c\right) \div d = \frac{ac}{b} \div d = \frac{ac}{bd}.$$

Hence, the product of two algebraic fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.

Illustrative Example. Multiply $\frac{x^2-1}{x^2-7x+10}$ by $\frac{x^2-3x+2}{x^2+2x+1}$ and reduce the resulting fraction to its lowest terms.

$$\frac{x^2-1}{x^2-7x+10} \times \frac{x^2-3x+2}{x^2+2x+1} = \frac{(x-1)(x+1)(x-2)(x-1)}{(x-2)(x-5)(x+1)(x+1)}$$

$$= \frac{(x-1)(x-1)}{(x-5)(x+1)} = \frac{x^2-2x+1}{x^2-4x-5}.$$

It is desirable to resolve each numerator and denominator into prime factors, and then cancel all common factors before performing any multiplication.

EXERCISES

Find the following indicated products and reduce each fraction to its lowest terms:

1. $\dfrac{3\,x^2y^2}{2\,yz^2} \times \dfrac{6\,az}{9\,x^3}.$

2. $\dfrac{5\,a\,(a-b)}{3\,c\,(a+b)} \times \dfrac{9\,(a+b)^2}{15\,(a^2-b^2)}.$

3. $\dfrac{12\,c^3b}{5\,(c^3-b^3)} \times \dfrac{35\,(c^2+cb+b^2)}{14\,c^2b^2}.$

4. $\dfrac{y^2+3\,y+2}{y^2-5\,y+6} \times \dfrac{y^2-7\,y+12}{y^2+8\,y+7}.$

5. $\dfrac{3^2\cdot 4^3}{5^2\cdot 2^4} \times \dfrac{10\cdot 2}{3^4}.$

6. $\dfrac{3^2\cdot 2^3}{5^2} \times \dfrac{5^4\cdot 7^2}{3^4\cdot 2^3} \times \dfrac{6\cdot 3^2}{5^4\cdot 7^3}.$

7. $\dfrac{x^2-x}{x^2-1} \times \dfrac{2\,x^2+4\,x+2}{3\,x^2+6\,x}.$

8. $\dfrac{a^2-10\,a+16}{a^2+6\,a+9} \times \dfrac{a+3}{a^2-4}.$

9. $\dfrac{(x+y)^2-z^2}{x^2+xy-xz} \times \dfrac{x}{(x-z)^2-y^2} \times \dfrac{(x-y)^2-z^2}{xy-y^2-yz}.$

10. $\dfrac{a^2+7\,a+12}{a^3+5\,a^2+6\,a} \times \dfrac{3\,a^3+27\,a^2+42\,a}{6\,a^2+66\,a+168}.$

11. $\dfrac{3^4(a-5)(a+2)}{2^3(a-3)(a-2)} \times \dfrac{2^2(a-3)(a-5)}{3^3(a-5)^2}.$

12. $\dfrac{3(x+4)^2}{4(x+4)(x-7)} \times \dfrac{(x-7)^2}{3(x+4)(x-7)}.$

13. $\dfrac{a^3+5\,a^2-36\,a}{a^2-7\,a-144} \times \dfrac{(a-16)(a-3)}{a(a-4)(a+2)}.$

14. $\dfrac{3\,a(a-7)(a-5)}{7\,b(a-3)(a-7)} \times \dfrac{b(a-3)(a+10)}{a(a-5)(a-10)}.$

15. $\dfrac{42\,(b-3)(b-4)}{3(b-4)(b+7)} \times \dfrac{6\,(b+7)(b-5)}{14(b-3)(b-6)}.$

16. $\dfrac{a(b^3-a^3)}{b(b+a)} \times \dfrac{(b^2-a^2)^2}{b^2+ba+a^2} \times \dfrac{(b+a)^2}{(b-a)^2}.$

17. $\dfrac{a^2-4a+3}{a^2-5a+4}\times\dfrac{a^2-9a+20}{a^2-10a+21}\times\dfrac{a^2-7a}{a^2-5a}.$

18. $\dfrac{3a(a-7)(a-5)}{7b(a-3)(a-7)}\times\dfrac{b(a-3)(a+10)}{a(a-5)(a-10)}.$

19. $\dfrac{a^2-12a+35}{c(a^2-10a+21)}\times\dfrac{a^2-8a+15}{a^2+3a-70}.$

20. $\dfrac{c^2-10c+21}{c^2-18c+77}\times\dfrac{c^2-3c-88}{c^2-8c+15}.$

21. $\dfrac{x^2(x^2-16)}{y^2(y^2-3y-28)}\times\dfrac{y^2-12y+35}{x^2-9x+20}\times\dfrac{y^2(x+5)}{x^3(x-4)}.$

22. $\dfrac{3a^2b^2(c^2-14c+33)}{4ab(c^2-10c+21)}\times\dfrac{2(c^2+c-56)}{a^2(c^2-16c+55)}\times\dfrac{c^2-2c-15}{b^2(c^2+12c+32)}.$

23. $\dfrac{3t^2-2t-1}{2t^2+t-1}\times\dfrac{2t^2+5t-3}{3t^2+7t+2}\times\dfrac{4t^2+10t+4}{4t^2-2t-2}.$

24. $\dfrac{6x^2-7x+2}{10x^2-7x+1}\times\dfrac{6x^2-5x-1}{6x^2+x-1}\times\dfrac{10x^2+3x-1}{5x^2-4x-1}.$

25. $\dfrac{4b^2-17b+4}{6b^2-7b+2}\times\dfrac{10b^2-21b+9}{5b^2-23b+12}\times\dfrac{3b^2-5b+2}{4b^2-5b+1}.$

TO DIVIDE A FRACTION BY A FRACTION

194. In arithmetic, to divide a number by the quotient of two numbers is the same as to divide by the dividend and multiply the result by the divisor.

E.g. $$72 \div \frac{18}{3} = 72 \div 6 = 12;$$

or, $$72 \div \frac{18}{3} = (72 \div 18)\cdot 3 = 4\cdot 3 = 12.$$

Likewise, $$\frac{2}{3}\div\frac{5}{7} = \left(\frac{2}{3}\div 5\right)\cdot 7 = \left(\frac{2}{3\cdot 5}\right)\cdot 7 = \frac{2\cdot 7}{3\cdot 5}.$$

Since in algebra a fraction is an indicated quotient, to divide a number by an algebraic fraction we divide by the numerator and multiply this result by the denominator.

E.g. $$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \div c\right) \times d = \left(\frac{a}{b \cdot c}\right) \times d = \frac{ad}{bc}.$$

Hence, in algebra, as in arithmetic, a number is divided by a fraction by inverting the fraction and multiplying by the new fraction thus obtained.

Thus, $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

EXERCISES

Perform the following indicated divisions, and reduce the resulting fractions to their lowest terms:

1. $\dfrac{a^3+b^3}{a^2-9b^2} \div \dfrac{a+b}{a+3b}$.

2. $\dfrac{x^2+x-2}{x^2-3x} \div \dfrac{x^3+2x^2}{x^2+9x-36}$.

3. $\dfrac{x^2-6x-16}{x^2+4x-21} \div \dfrac{x^2+9x+14}{x^2-8x+15}$.

4. $\dfrac{x^2-1}{x^2-4x-5} \div \dfrac{x^2+2x-3}{x^2-25}$.

5. $\dfrac{a^2-11a-26}{a^2-3a-18} \div \dfrac{a^2-18a+65}{a^2-9a+18}$.

6. $\dfrac{x^2+9xy+18y^2}{x^2-9xy+20y^2} \div \dfrac{x^2+6xy+9y^2}{xy^2-4y^3}$.

7. $\dfrac{x^2+mx+nx+mn}{x^2-mx-nx+mn} \div \dfrac{x^2-n^2}{x^2-m^2}$.

8. $\dfrac{3a^4-9a^3-54a^2}{9a^3-117a^2+378a} \div \dfrac{a^3+8a^2+15a}{3a^2-33a+84}$.

9. $\dfrac{a^2-11a+30}{a^3-6a^2+9a} \times \dfrac{a^2-3a}{a^2-25} \div \dfrac{a^2-9}{a^2+2a-15}$.

10. $\dfrac{x^2-10x+21}{x^2+x-56}\div\dfrac{x^2-8x+15}{x^2+4x-32}$.

11. $\dfrac{a^2-3a+2}{a^2-7a+12}\times\dfrac{a^2-8a+15}{a^2-6a+5}\div\dfrac{a^2-9a+14}{a^2-12a+32}$.

12. $\dfrac{n^2-11n+18}{m^2-7m-18}\times\dfrac{n^2-8n-9}{m^2-5m-14}\div\dfrac{6n-12}{an+a}$.

13. $\dfrac{a^2-b^2}{ab^2x}\times\dfrac{b(a-b)}{a^2+2ab+b^2}\div\dfrac{b(a+b)}{a^2-2ab+b^2}$.

14. $\dfrac{a+b}{ab}\times\dfrac{a^2-b^2}{3(a^2+b^2)}\div\dfrac{(a-b)^2(a+b)^2}{3a^3y+3ay^3}$.

15. $\dfrac{5x^4-5x^3}{7x^2-56x-63}\div\dfrac{x^4-9x^3+8x^2}{14x^2+14x-1260}$.

16. $\dfrac{8y^2(y+4)(y+5)}{2^4(y+5)(y-7)}\div\dfrac{y(y+4)(y+8)}{2^3(y-7)(y+11)}$.

17. $\dfrac{(c+4)(c-4)}{(c-3)(c-2)}\times\dfrac{(c+2)(c-5)}{(c-5)(c-6)}\div\dfrac{(c+4)(c-13)}{(c-6)(c+11)}$.

18. $\dfrac{2a^2b(b-4)(b+4)}{b(b+4)(b+6)}\times\dfrac{(b+6)(b+8)}{(b+8)(b-9)}\div\dfrac{a(b+8)(b-4)}{(b-9)(b-5)}$.

19. $\dfrac{x^3(x-2)^2}{(x+2)^2}\times\dfrac{(x+2)(x-3)}{(x-2)(x-7)}\div\dfrac{x^2(x-3)(x-5)}{(x-5)(x-7)}$.

20. $\dfrac{a^2b^2(c+5)(c-4)}{(c-4)(c-8)}\times\dfrac{(c-8)(c+9)}{(c+4)(c+7)}\div\dfrac{ab(c+9)(c-1)}{(c+7)(c+1)}$.

21. $\dfrac{21x^2+23x-20}{10x^2-27x+5}\times\dfrac{6x^2-11x-10}{3x^2+2x-5}\div\dfrac{7x^2+17x-12}{5x^2+9x-2}$.

22. $\dfrac{x^2+6x+9}{9x^2-4}\times\dfrac{9x^2+9x-10}{7x^2+20x-3}\div\dfrac{3x^2-7x-20}{35x^2-12x+1}.$

23. $\dfrac{(a+4)^2-b^2}{12x^3-16x^2}\times\dfrac{3x-4}{ax^2+4x^2-bx^2}\div\dfrac{ab^2+4b^2+b^3}{x^3}.$

24. $\dfrac{12yx^2-18yx}{15x^2+32x-7}\times\dfrac{3bx^2+7bx}{6xy+15y}\div\dfrac{3bx^3+7bx^2}{10x^2+23x-5}.$

25. $\dfrac{ac+bc-ar-br}{3a^2-8a-3}\times\dfrac{6a^2+23a+7}{cb-rb+c^2-rc}\div\dfrac{3a^2+a+b+3ab}{ab-3b+ac-3c}.$

COMPLEX FRACTIONS

195. Sometimes fractions occur whose numerators or denominators, or both, contain fractions.

E.g.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}} \text{ and } \frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}}.$$

Such fractions are called **complex fractions**. A complex fraction is said to be **simplified** when it is reduced to an equal fraction whose numerator and denominator are in the integral form.

Ex. 1.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}}=\frac{\frac{a}{a}+\frac{1}{a}}{\frac{a}{a}-\frac{1}{a}}=\frac{a+1}{a}\div\frac{a-1}{a}$$
$$=\frac{a+1}{a}\times\frac{a}{a-1}=\frac{a+1}{a-1}.$$

This result may also be obtained directly by multiplying both terms of the given fraction by a, finding at once $\dfrac{a+1}{a-1}$.

Ex. 2.
$$\frac{\dfrac{1}{x+1}+\dfrac{1}{x-1}}{\dfrac{1}{x-1}-\dfrac{1}{x+1}}=\frac{\dfrac{x-1+x+1}{x^2-1}}{\dfrac{x+1-x+1}{x^2-1}}$$

$$=\frac{2x}{x^2-1}\times\frac{x^2-1}{2}=x.$$

By multiplying the terms of the given fraction by $(x+1)(x-1)$ we may also get directly $\dfrac{x-1+x+1}{x+1-x+1}=x.$

Such fractions seldom occur in the problems of elementary algebra. A more extended discussion is given in the Advanced Course.

EXERCISES

Reduce each of the following complex fractions to its simplest form:

1. $\dfrac{1+x}{1+\dfrac{1}{x}}$.

2. $\dfrac{1-\dfrac{a}{b}}{1+a}$.

3. $\dfrac{x+\dfrac{x}{2}}{x-\dfrac{x}{2}}$.

4. $\dfrac{\dfrac{m+n+1}{3}}{\dfrac{m-n-1}{2}}$.

5. $\dfrac{4+\dfrac{a+b}{2}}{4-\dfrac{a-b}{2}}$.

6. $\dfrac{\dfrac{x^2-y^2}{4}}{\dfrac{x+y}{2}}$.

7. $\dfrac{\dfrac{a^3-8b^3}{27}}{3a-2b}$.

8. $\dfrac{\dfrac{M}{D^3}}{\dfrac{1+d^2}{D^2}}$.

9. $\dfrac{\dfrac{H}{c}-\dfrac{hd}{cD}}{\dfrac{1+t}{c}}$.

RATIO AND PROPORTION

196. Definitions. A fraction is often called a **ratio.** Thus $\frac{a}{b}$ may be read *the ratio of a to b,* and is also written $a : b$.

The numerator is called the **antecedent** of the ratio, and the denominator the **consequent.** The antecedent and consequent are called the **terms** of the ratio.

An equation, each of whose members is a ratio, is called a **proportion.**

Thus, $\frac{a}{b}=\frac{c}{d}$ is a proportion, and is also written $a : b = c : d$.

It is read *the ratio of a to b equals the ratio of c to d,* or briefly, *a is to b as c is to d.*

The four numbers a, b, c, and d, are said to be **in proportion.** a and d are called the **extremes** of the proportion, and b and c the **means.**

IMPORTANT PROPERTIES OF A PROPORTION

1. If, in the proportion $\frac{a}{b}=\frac{c}{d}$, both members of the equation be multiplied by bd, we have, $ad = bc$.

That is: *If four numbers are in proportion, the product of the means equals the product of the extremes.*

2. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

Hint. Divide $1 = 1$ by the members of the given equation.

This process is called taking the proportion by **inversion.**

3. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

Hint. Multiply both members of the given equation by $\frac{b}{c}$.

This process is called taking the proportion by **alternation.**

4. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Hint. Add 1 to both members of the given equation.

This process is called taking the proportion by **composition.**

5. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

Hint. Subtract 1 from each member of the given equation.

This process is called taking the proportion by **division.**

6. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Hint. Divide the members of the equation obtained under 4 by the members of the one obtained under 5.

7. Show that if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk,\ c=dk,\ e=fk.$

Hence, $a+c+e=bk+dk+fk=(b+d+f)k,$

and $$\frac{a+c+e}{b+d+f}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

That is, *If several ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

EXERCISES

1. If $ad=bc$, show that $\frac{a}{b}=\frac{c}{d}$. *Hint.* Divide by bd.

2. If $ad=bc$, show that $\frac{a}{c}=\frac{b}{d}$.

3. If $ad=bc$, show that $\frac{d}{c}=\frac{b}{a}$.

4. If $ad = bc$, show that $\frac{d}{b} = \frac{c}{a}$.

5. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$.

6. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a} = \frac{c-d}{c}$.

7. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a-b}{c-d}$.

9. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a}{c}$.

10. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+c}{b+d} = \frac{a}{b}$.

11. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{c-d} = \frac{a}{c}$.

12. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-c}{b-d} = \frac{a}{b}$.

13. Solve the equation $\frac{a}{b} = \frac{c}{d}$ for each letter in terms of all the others. If $a = 3$, $b = 5$, $c = 8$, find d. If $b = 7$, $c = 9$, $d = 3$, find a. If $c = 13$, $d = 2$, $a = 5$, find b. If $d = 50$, $a = 3$, $b = -7$, find c.

14. If $\frac{a}{b} = \frac{c}{x}$, then x is said to be a **fourth proportional** to a, b, and c.

Find a fourth proportional to 3, 5, and 7; also to 9, 5, and 1, and to 3, -2, and -5.

15. If $\frac{a}{x}=\frac{x}{b}$, then x is called a **mean proportional** between a and b.

Solve the equation $\frac{a}{x}=\frac{x}{b}$ for x in terms of a and b. Show that there are two solutions, each of which is a mean proportional between a and b.

Find two mean proportionals between 4 and 9; also between 5 and 125, and between -4 and -36.

16. Which is the greater ratio $\frac{5+3d}{5+4d}$ or $\frac{5+4d}{5+5d}$.

Hint. Reduce the fractions to a common denominator and compare numerators. (d is a positive number.)

17. Which is the greater ratio, $\frac{a+7b}{a+8b}$ or $\frac{a+9b}{a+10b}$.

18. Which is the greater ratio, $\frac{a}{b}$ or $\frac{a+c}{b+c}$, if b and c are positive, and a less than b? a equal to b? a greater than b?

19. Find two numbers in the ratio of 3 to 5 whose sum is 160.

20. Find two numbers in the ratio of 2 to 7 whose sum is -108.

21. Find two numbers in the ratio of 3 to -4 whose sum is -15.

22. What number added to each of the terms of the ratio $\frac{5}{7}$ makes it equal to $\frac{65}{75}$?

23. What number must be added to each term of the ratio $\frac{1}{11}$ to make it equal to the ratio $\frac{3}{5}$?

24. What number added to each of the numbers 3, 5, 7, 10, will make the sums in proportion, when taken in the given order?

25. Two numbers are in the ratio of 2 to 3, and the sum of their squares is 325. Find the numbers.

EQUATIONS INVOLVING FRACTIONS

197. We have already seen, § 40, that in solving an equation involving fractions the first step is to multiply both members of the equation by a number which will cancel all the denominators. Evidently, any common multiple of the denominators is such a multiplier. For the sake of simplicity, and for reasons which are fully discussed in the Advanced Course, the *lowest common multiple is always used for this purpose.*

Illustrative Example. Solve the equation:

$$\frac{x}{4}+\frac{x-6}{6}-\frac{1-2x}{8}=\frac{55}{8}. \tag{1}$$

Solution. The L.C.M. of 4, 6, and 8 is 24. Multiplying both members of the equation by 24,

$$\frac{24x}{4}+\frac{24(x-6)}{6}-\frac{24(1-2x)}{8}=\frac{24\cdot 55}{8}. \tag{2}$$

By *F*, *V*, $$6x+4(x-6)-3(1-2x)=3\cdot 55=165. \tag{3}$$

By *F*, $$6x+4x-24-3+6x=165. \tag{4}$$

By *F*, *A*, $$16x=192. \tag{5}$$

By *D*, $$x=12. \tag{6}$$

As in this solution, so in general, each denominator is canceled by Principle V, after multiplying by the L. C. M. In practice, however, equation (2) may be omitted, the canceling being done mentally, and equation (3) may be written down at once..

The dividing line of the fraction acts like a parenthesis with respect to the sign preceding the fraction.

EXERCISES

Solve the following equations, and check each solution by substituting in the original equation:

1. $\dfrac{4x-5}{3}-\dfrac{2x+5}{5}=\dfrac{x-3}{2}+\dfrac{7x+5}{8}-4.$

2. $\dfrac{x+3}{2}-\dfrac{5x-3}{2x}+\dfrac{5x-3}{4}=4.$

3. $\dfrac{2x+5}{3}=\dfrac{5x+20}{15}+\dfrac{7x+13}{2}+\dfrac{15x+3}{4}.$

4. $\dfrac{3x+10}{5}-\dfrac{7x+15}{3}+\dfrac{5x-14}{7}=\dfrac{3x-12}{4}-2.$

5. $x+\dfrac{3x+5}{4x}+\dfrac{7-3x}{2x}=\dfrac{15-x}{2x}+\dfrac{3-11x}{4}.$

6. $\dfrac{7x-2}{6}-\dfrac{3x-24}{9}=\dfrac{3x+4}{2}+\dfrac{5-4x}{3}.$

7. $\dfrac{2x}{3}-\dfrac{5x}{12}+\dfrac{7x}{8}-\dfrac{x}{2}=45.$

8. $\dfrac{2x}{3}-\dfrac{3x}{4}+\dfrac{5x}{7}-\dfrac{11x}{12}=-24.$

9. $\dfrac{s-3}{2s}+\dfrac{2s+3}{3}-\dfrac{5s-3}{6}=\dfrac{3s-1}{2}-3.$

10. $\dfrac{2t+1}{5}-\dfrac{t-1}{8}+\dfrac{4t-8}{15}=\dfrac{3t-1}{10}+3.$

11. $\dfrac{x+3}{2}-\dfrac{4x+5}{7}+\dfrac{3x-5}{4}=\dfrac{5x-7}{12}+3.$

12. $\frac{8k-6}{5}+\frac{13k}{2}-\frac{21k-12}{5}=\frac{k-2}{10k}+\frac{14k-3}{5}+4.$

13. $\frac{5x-1}{2}-\frac{2x+3}{3}-\frac{5x+1}{4}=\frac{13x+5}{11}-4.$

14. $2-\frac{h-15}{2}+\frac{h-20}{3}-\frac{4h+2}{11}=0.$

15. $-\frac{17+m}{4}-\frac{2m-7}{3}+\frac{5m-3}{2}+2=\frac{3}{m}.$

16. $\frac{3r+4}{5}-\frac{5r+1}{12}-\frac{8r+4}{10}=2-\frac{6r}{7}.$

17. $\frac{9+2y}{3}-\frac{4y+3}{3y}=\frac{27+3y}{6}-3.$

18. $\frac{2x+5}{3}-\frac{5x+2}{7}+\frac{4x-5}{9}=\frac{3x+4}{7}.$

19. $\frac{z+3}{2}-\frac{2z-15}{z}-\frac{5z-11}{8}-\frac{11}{2}=0.$

20. $\frac{3x+20}{2}+\frac{5x-3}{11}-\frac{4x-1}{5}=3.$

198. Illustrative Example. Solve the following equation:

$$\frac{2x-1}{x-1}+\frac{4}{x+1}-\frac{3x}{x^2-1}=2. \qquad (1)$$

Solution. The L. C. M. of the denominators is x^2-1. In multiplying both members of the equation by x^2-1, $x-1$ is canceled in the first fraction, $x+1$ in the second, and x^2-1 in the third, giving

$$(2x-1)(x+1)+4(x-1)-3x=2(x^2-1) \qquad (2)$$

Solving, $$2x^2+x-1+4x-4-3x=2x^2-2, \qquad (3)$$

$$2x=3, \qquad (4)$$

and $$x=\tfrac{3}{2}. \qquad (5)$$

Check by substituting $x=\frac{3}{2}$ in (1).

EXERCISES

Solve the following equations and check each solution by substituting in the original equation:

1. $\frac{3x-1}{x+1}-\frac{4x+3}{x-1}+\frac{x^2}{x^2-1}=-\frac{27}{x^2-1}+1.$

2. $\frac{3x+5}{x-9}+\frac{2x+1}{x+2}=\frac{x-1}{x^2-7x-18}.$

3. $\frac{3x-4}{x+5}-\frac{4x-1}{x+4}+\frac{x^2+44}{x^2+9x+20}=0.$

4. $\frac{4-x^2}{x^2-5x-14}-\frac{3x+6}{x+2}=-\frac{2x+1}{x-7}.$

5. $\frac{x-4}{2x-10}-\frac{3x-15}{2x-6}=-\frac{3x^2-114}{4x^2-32x+60}.$

6. $\frac{6(x+4)}{x+5}-\frac{3(2x-1)}{x+1}=\frac{7}{2}.$

7. $\frac{3x-4}{x-4}+\frac{5x-7}{2x-2}=\frac{9x^2-38}{2x^2-10x+8}.$

8. $\frac{x+17}{x+5}-\frac{2(x+6)}{x+3}=-\frac{x-1}{x+3}.$

9. $\frac{x+2}{x-5}+\frac{3x-15}{x-3}=\frac{3x-21}{x-3}.$

10. $\frac{2x-3}{-4x}+\frac{3x+1}{x-2}=\frac{4x+17}{x-2}.$

11. $\dfrac{3x-2}{2x+3}=\dfrac{2x^2+15x+28}{2x^2+5x+3}+\dfrac{2x-1}{x+1}$.

12. $\dfrac{2x-3}{2x+2}-\dfrac{x-8}{5x+2}=\dfrac{x+2}{2x+2}$.

13. $\dfrac{20x^2+7x-3}{9x^2-1}-\dfrac{3x+1}{3x-1}=1$.

14. $\dfrac{7x^2+11x+4}{6x^2+13x+5}+\dfrac{x+3}{2x+1}=\dfrac{7x+11}{3x+5}$.

15. $\dfrac{3x+1}{5x-7}-\dfrac{x-3}{2x-7}=\dfrac{2x^2-10x+12}{10x^2-49x+49}$.

199. Sometimes it is best to *add fractions before multiplying* by the L. C. M. and in other cases to multiply by the L. C. M. of *part of the denominators first*, and, after simplifying, multiply by the L. C. M. of the remaining denominators.

Ex. 1. Solve the equation

$$\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3}. \qquad (1)$$

Adding fractions on the right and left,

$$\frac{1}{(x-2)(x-1)}=\frac{1}{(x-4)(x-3)}. \qquad (2)$$

Multiplying by L. C. M., $(x-4)(x-3)=(x-2)(x-1)$. (3)

Hence $4x=10$, (4)

and $x=2\frac{1}{2}$. (5)

Check by substituting $x=2\frac{1}{2}$ in equation (1).

Ex. 2. Solve the equation:

$$\frac{4t-3}{16}-\frac{t-2}{4}=\frac{2t-2}{5t+2}. \qquad (1)$$

Multiplying by 16, $$4t-3-4t+8=\frac{32t-32}{5t+2}. \qquad (2)$$

Hence, $$5=\frac{32t-32}{5t+2}. \qquad (3)$$

Multiplying by $5t+2$, $$25t+10=32t-32. \qquad (4)$$

Hence, $$7t=42, \qquad (5)$$

and $$t=6. \qquad (6)$$

Check by substituting $t=6$ in equation (1).

EXERCISES

1. $\dfrac{3x+6}{5}-\dfrac{9x+3}{15}=\dfrac{x+7}{6x-8}+4.$

2. $\dfrac{7x+1}{12}-\dfrac{14x-22}{24}=\dfrac{11x+5}{8x-28}.$

3. $\dfrac{3x+4}{2}-\dfrac{12x+1}{8}=\dfrac{5x-1}{3x+2}.$

4. $\dfrac{7t+3}{5}-\dfrac{21t+9}{15}=\dfrac{17t-3}{3t+11}+2.$

5. $\dfrac{11v-15}{10}-\dfrac{33v+15}{30}=\dfrac{5v+5}{v-5}.$

6. $\dfrac{x-1}{x-2}+\dfrac{x-2}{3-x}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

7. $\dfrac{1}{x-1}-\dfrac{2}{2x+1}=\dfrac{1}{x-2}-\dfrac{4}{4x+1}.$

8. $\dfrac{x-2}{x-3}-\dfrac{x-3}{x-4}=\dfrac{x-4}{x-5}+\dfrac{x-5}{6-x}.$

9. $\dfrac{9}{x-7}-\dfrac{9}{x-2}=\dfrac{5}{x-8}-\dfrac{5}{x+1}.$

10. $\dfrac{x-1}{x-2}+\dfrac{x-2}{3-x}=\dfrac{x-3}{x-4}-\dfrac{x-5}{x-6}.$

SIMULTANEOUS FRACTIONAL EQUATIONS

200. When pairs of fractional equations are given, each should be reduced to the integral form before eliminating, except in special cases like those in the second following illustrative example.

Ex. 1. Solve the equations:

$$\begin{cases} \dfrac{4}{x-y}+\dfrac{6}{x+y}=\dfrac{36}{x^2-y^2}. & (1) \\ \dfrac{3}{2x-y}-\dfrac{2}{x-3y}=\dfrac{-18}{(2x-y)(x-3y)}. & (2) \end{cases}$$

From (1) by M, $4(x+y)+6(x-y)=36.$ (3)

By F, D, $5x-y=18.$ (4)

From (2) by M, $3(x-3y)-2(2x-y)=-18.$ (5)

By F, D, $x+7y=18.$ (6)

From (4) by M, $35x-7y=126.$ (7)

Adding (6) and (7), $36x=144.$ (8)

By D, $x=4.$ (9)

Substitute $x=4$ in (6), $y=2.$ (10)

Check by substituting $x=4$, $y=2$ in (1) and (2).

Ex. 2. Solve the equations:

$$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=2, & (1) \\ \dfrac{20}{x}-\dfrac{21}{y}=3. & (2) \end{cases}$$

In this case it is best to solve the equations for $\dfrac{1}{x}$ and $\dfrac{1}{y}$ instead of for x and y.

From (1) by M, $$\frac{14}{x}+\frac{21}{y}=14. \qquad (3)$$

Adding (2) and (3), $$\frac{34}{x}=17. \qquad (4)$$

Hence by D, $$\frac{1}{x}=\frac{1}{2}. \qquad (5)$$

Substituting $\dfrac{1}{x}=\dfrac{1}{2}$ in (1), $$\frac{1}{y}=\frac{1}{3}. \qquad (6)$$

From (5) and (6) by M, $$x=2,\ y=3. \qquad (7)$$

EXERCISES

Solve the following equations:

1. $$\begin{cases} \dfrac{2x-1}{x+1}-\dfrac{3y-1}{y+1}=\dfrac{-xy}{(x+1)(y+1)}, \\ \dfrac{x+2}{2y-1}+\dfrac{2x-1}{y+1}=\dfrac{5xy}{(2y-1)(y+1)}. \end{cases}$$

2. $$\begin{cases} \dfrac{3x+2}{3y-5}=\dfrac{x+1}{y-1}, \\ \dfrac{3x-2}{y+1}=\dfrac{3x-1}{y-1}-\dfrac{2}{(y-1)(y+1)}. \end{cases}$$

3. $\begin{cases} \dfrac{5}{t}-\dfrac{6}{v}=2, \\ \dfrac{17}{v}+\dfrac{4}{t}=67. \end{cases}$

4. $\begin{cases} \dfrac{3}{x}+\dfrac{1}{y}=21, \\ \dfrac{7}{x}-\dfrac{9}{y}=-19. \end{cases}$

5. $\begin{cases} \dfrac{7}{a}-\dfrac{1}{b}=12\frac{1}{2}, \\ \dfrac{3}{a}+\dfrac{12}{b}=24. \end{cases}$

6. $\begin{cases} \dfrac{3}{x}-\dfrac{2}{y}=-4, \\ \dfrac{6}{x}+\dfrac{11}{y}=52. \end{cases}$

7. $\begin{cases} \dfrac{12}{x}-\dfrac{10}{y}=1, \\ \dfrac{9}{x}+\dfrac{2}{y}=15. \end{cases}$

8. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=1, \\ \dfrac{c}{x}+\dfrac{d}{y}=1. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=16, \\ \dfrac{1}{y}+\dfrac{1}{z}=14, \\ \dfrac{1}{z}+\dfrac{1}{x}=12. \end{cases}$

10. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{y}+\dfrac{1}{z}=b, \\ \dfrac{1}{z}+\dfrac{1}{x}=c. \end{cases}$

In 9 first add all three equations, and from half the sum subtract each equation separately. Likewise in 10.

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

In solving the following problems use one or two unknowns as may be found most convenient.

1. There are two numbers whose sum is 51 such that if the greater is divided by their difference, the quotient is $3\frac{1}{8}$. Find the numbers.

2. There are two numbers whose sum is 91 such that if the greater is divided by their difference, the quotient is 7. Find the numbers.

3. There are two numbers whose sum is s such that if the greater is divided by their difference, the quotient is q. Find an expression in terms of s and q representing each number. Solve 1 and 2 by substituting in the formula just obtained.

4. What number must be subtracted from each term of the fraction $\frac{11}{17}$ so that the result shall be equal to $\frac{1}{2}$?

5. What number must be subtracted from each term of the fraction $\frac{21}{35}$ so that the result shall be equal to $\frac{2}{9}$?

6. What number must be subtracted from each term of the fraction $\dfrac{a}{b}$ so that the result shall be equal to $\dfrac{c}{d}$? Solve 4 and 5 by substituting in the formula obtained under 6.

7. What number must be added to each term of the fraction $\frac{1}{2}$ to obtain a fraction equal to $\frac{16}{17}$?

8. What number must be added to each term of the fraction $\dfrac{a}{b}$ to obtain a fraction equal to $\dfrac{c}{d}$?

By means of the formula thus obtained, solve problem 7, and also 4, 5, and 6. (See note under Problem 23, p. 114.)

9. There are two numbers whose difference is 153. If their sum is divided by the smaller, the quotient is equal to $\frac{47}{15}$. Find the numbers.

10. There are two numbers whose difference is b. If their sum is divided by the smaller, the quotient is q. Find the numbers. Solve 9 by substituting in this formula.

11. Divide 548 into 2 parts, such that 7 times the first shall exceed 3 times the second by 474.

12. There are two numbers whose sum is 48 such that 3 times the first is 8 more than 5 times the second. Find both numbers.

13. There are two numbers whose sum is s such that a times the first is b more than c times the second. Find both numbers.

14. What number must be subtracted from each of the numbers 12, 15, 19, and 25 in order that the remainders may form a proportion when taken in the given order?

15. What number must be added to each of the numbers 13, 21, 3, and 8 so that the sums shall be in proportion when taken in the order given?

16. What number must be added to each of the numbers a, b, c, d so that the sums shall be in proportion when taken in the given order?

17. What number must be subtracted from each of the numbers a, b, c, d so that the remainders shall be in proportion when taken in the given order?

Compare the results in 16 and 17 and explain the relation between them. (See remark under Problem 23, page 114.)

Solve 14 and 15 by substituting in the formulas obtained in 16 and 17.

18. There is a number composed of two digits whose sum is 11. If the number is divided by the difference between the digits, the quotient is $16\frac{4}{5}$. Find the number, the tens' digit being the larger.

19. There is a number composed of two digits whose sum is s. If the number is divided by the difference between the digits, the quotient is q. Find the number, the tens' digit being the larger.

20. **Illustrative Problem.** A can do a piece of work in 8 days, B can do it in 10 days. In how many days can they do it together?

Since A can do the work in 8 days, in one day he can do $\frac{1}{8}$ of it, and since B can do it in 10 days, in one day he can do $\frac{1}{10}$ of it. If x is the number of days required when both work together, in one day they can do $\frac{1}{x}$ of it. Hence we have the equation,

$$\frac{1}{8}+\frac{1}{10}=\frac{1}{x}.$$

21. A can do a piece of work in 12 days and B can do it in 9 days. How long will it take both working together to do it?

22. A pipe can fill a cistern in 11 hours and another in 13 hours. How long will it require both pipes to fill it?

23. A can do a piece of work in a days and B can do it in b days. How long will it take both together to do it?

24. A cistern can be filled by one pipe in 20 minutes and emptied by another in 30 minutes. How long will it take to fill the cistern when both are running together?

25. A pipe can fill a cistern in 12 hours, another in 10 hours, and a third can empty it in 8 hours. How long will it require to fill the cistern when they are all running?

26. A man can do a piece of work in 18 days, another in 21 days, a third in 24 days, and a fourth in 10 days. How long will it require them when all are working together?

27. A and B working together can do a piece of work in 12 days. B and C working together can do it in 13 days, and A and C working together can do it in 10 days. How long will it require each to do it when working alone?

Suggestion: Let $a =$ the fraction of the work A can do in one day, $b =$ the fraction of the work B can do in one day, and $c =$ the fraction of the work C can do in one day.

Then $\quad a + b = \frac{1}{12}, \quad b + c = \frac{1}{13}, \quad c + a = \frac{1}{10}.$

28. A and B working together can do a piece of work in l days. B and C can do it in m days and C and A can do it in n days. How long will it require each working alone?

29. The circumference of the rear wheel of a carriage is 1.8 feet more than that of the front wheel. In running one mile the front wheel makes 48 revolutions more than the rear wheel. Find the circumference of each wheel.

If x is the number of feet in the circumference of the front wheel, then $\frac{5280}{x}$ is the number of revolutions in going one mile.

30. The circumference of the rear wheel of a carriage is 1 foot more than that of the front wheel. In going one mile the two wheels together make 920 revolutions. Find the circumference of each.

31. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour the running time is decreased by 2 hours. Find the speed of the train.

If r is the original rate of the train, then $\frac{420}{r}$ is the running time.

32. The distance from New York to Buffalo is 442 miles. By decreasing the speed of a fast freight 8 miles per hour the running time is increased 4 hours. Find the speed of the freight.

33. A motor boat goes 10 miles per hour in still water. In 10 hours the boat goes 42 miles up a river and back again. What is rate of the current?

34. A train leaving New York over the Pennsylvania Road requires 9 hours to overtake a train leaving Philadelphia westward at the same time. If the Philadelphia train had started toward New York, they would have met in one hour. Find the rate of each train, the distance from New York to Philadelphia being 90 miles.

35. The average of the numbers x, 34, 0, -58, -19, 0, -20, and y is 12; while the average of $2x$, $3y$, -18, 50, and -30 is -4. Find x and y.

36. The length of a rectangle is 8 feet greater, and its width is 4 feet greater, than the side of a certain square. The sum of the areas of the square and rectangle is 736 square feet. Find the dimensions of each.

37. The difference between the areas of a circle and its circumscribed square is 12 square inches. Find the radius of the circle. (See problem 33, p. 239.)

38. The difference between the areas of a circle and its inscribed square is 12 square inches. Find the radius of the circle.

39. The difference between the areas of a circle and the regular inscribed hexagon is 12 square inches. Find the radius of the circle.

40. The altitude of an equilateral triangle is 6. Find its side and also its area. Find the side and area, if the altitude is h.

41. The radius of a circle is 3 feet. Find the area of the regular circumscribed hexagon. Find the area if the radius is r feet.

42. The radius of a circle is r. Find the difference between the areas of the circle and the regular circumscribed hexagon.

43. The difference between the areas of a circle and the regular circumscribed hexagon is 9 square inches. Find the radius of the circle.

44. A circle is inscribed in a square and another circumscribed about it. The area of the ring formed by the two circles is 25 square inches. How long is the side of the square?

45. A square is inscribed in a circle and another circumscribed about it. The area of the strip inclosed by the two squares is 25 square inches. Find the radius of the circle.

In the following five problems solve the resulting literal equations for each letter in terms of the others, and for each solution state a corresponding problem.

46. A hound pursuing a deer gains 400 yards in 25 minutes. If the deer runs 1300 yards a minute, how fast does the hound run? If the hound gains v_1 yards in t minutes and the deer runs v_2 yards per minute, find the speed of the hound.

47. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet? If the steamer is s knots from port and making v_1 knots per hour, and if the tug makes v_2 knots per hour, find how long before they will meet.

48. A motor boat starts $7\frac{3}{8}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How far apart will they be after sailing $1\frac{1}{3}$ hours? If the motor boat starts s miles behind the sailboat and runs v_1 miles per hour, while the sailboat runs v_2 miles per hour, how far apart will they be in t hours?

49. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one such position? If the liner makes v_1 knots per hour and the freight boat, which is s_1 knots out, makes v_2 knots per hour, how long before they will be s_2 knots apart?

50. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the distance is 500 miles, in how many hours will they meet? If they meet in 8 hours, how long is the road? If the rates of the trains are v_1 and v_2 and the road is s miles long, find the time.

51. In going 1200 yards the rear wheel of a carriage makes 60 revolutions less than the front wheel. If the circumference of each wheel be increased by 3 feet, the rear wheel will make only 40 revolutions less than the front wheel. Find the circumference of each wheel.

Part II

ADVANCED COURSE

HIGH SCHOOL ALGEBRA

ADVANCED COURSE

CHAPTER I

FUNDAMENTAL LAWS

1. We have seen in the Elementary Course that **algebra,** like arithmetic, deals with **numbers** and with operations upon numbers. We now proceed to study in greater detail the laws that underlie these operations.

THE AXIOMS OF ADDITION AND SUBTRACTION

2. In performing the elementary operations of algebra we assume at the outset certain simple statements called **axioms.**

Definition. Two number expressions are said to be equal if they represent the same number.

Axiom I. *If equal numbers are added to equal numbers, the sums are equal numbers.*

That is, if $a = b$ and $c = d$, then $a + c = b + d$.

Axiom I implies that *two numbers have one and only one sum.* This fact is often referred to as the **uniqueness of addition.**

3. If $a = c$ and $b = c$ then $a = b$, since the given equations assert that a is the same number as b. Hence the usual statement: *If each of two numbers is equal to the same number, they are equal to each other.*

4. The sum of two numbers, as 6 and 8, may be found by adding 6 to 8 or 8 to 6, in either case obtaining 14 as the result.

This is a particular case of a general law for all numbers of algebra, which we state as

Axiom II. *The sum of two numbers is the same in whatever order they are added.*

This is expressed in symbols by the identity:

$$a + b \equiv b + a. \qquad \text{[See § 37, E. C.*]}$$

Axiom II states what is called the **commutative law of addition**, since it asserts that numbers to be added may be *commuted* or interchanged in order.

Definition. Numbers which are to be added are called **addends.**

5. In adding three numbers such as 5, 6, and 7 we first add two of them and then add the third to this sum. It is immaterial whether we first add 5 and 6 and then add 7 to the sum, or first add 6 and 7 and then add 5 to the sum. This is a particular case of a general law for all numbers of algebra, which we state as

Axiom III. *The sum of three numbers is the same in whatever manner they are grouped.*

In symbols we have $\quad a + b + c \equiv a + (b + c).$

When no symbols of grouping are used, we understand $a + b + c$ to mean that a and b are to be added first and then c is to be added to the sum.

Axiom III states what is called the **associative law of addition,** since it asserts that addends may be *associated* or grouped in any desired manner.

It is to be noted that an equality may be read in either direction. Thus $\quad a + b + c = a + (b + c)$ and $a + (b + c) = a + b + c$ are equivalent statements.

6. If any two numbers, such as 19 and 25, are given, then in arithmetic we can always find a number which added to the smaller gives the larger as a sum. That is, we can subtract the smaller number from the larger.

* E. C. means the Elementary Course.

In algebra, where negative numbers are used, any number may be subtracted from any other number.

That is: *For any pair of numbers* a *and* b *there is one and only one number* c *such that* $a+c=b$.

The process of finding the number c when a and b are given is called **subtraction**. This operation is indicated thus, $b-a=c$, where b is the **minuend**, a the **subtrahend**, and c the **remainder**.

Since for a given minuend and a given subtrahend there is one and only one remainder, we have for all numbers of Algebra,

Axiom IV. *If equal numbers are subtracted from equal numbers, the remainders are equal numbers.*

Definitions. If $a+c=a$, then the number c is called **zero**, and is written 0. That is, $a+0=a$, or $a-a=0$. Hence zero is the remainder when minuend and subtrahend are equal.

By definition of subtraction, the equality $b-a=c$ implies that c is a number such that $c+a=b$.

Adding a to each member of the equality $b-a=c$, we have $b-a+a=c+a$, which by hypothesis is equal to b. Hence *subtracting a number and then adding the same number gives as a result the original number operated upon.*

Axiom IV implies the **uniqueness of subtraction.**

THE AXIOMS OF MULTIPLICATION AND DIVISION

7. Axioms similar to those just given for addition and subtraction hold for multiplication and division.

Axiom V. *If equal numbers are multiplied by equal numbers, the products are equal numbers.*

This axiom implies the **uniqueness of multiplication.** That is, *two numbers have one and only one product.*

8. The product of 5 and 6 may be obtained by taking 5 six times, or by taking 6 five times. That is, $5\cdot 6=6\cdot 5$. This is a special case of a general law for all numbers of algebra, which we state as

Axiom VI. *The product of two numbers is the same in whatever order they are multiplied.*

In symbols we have $\boldsymbol{a \cdot b \equiv b \cdot a}$.

This axiom states what is called the **commutative law of factors** in multiplication.

9. The product of three numbers, such as 5, 6, and 7, may be obtained by multiplying 5 and 6, and this product by 7, or 6 and 7, and this product by 5. This is a special case of a general law for all numbers of algebra, which we state as

Axiom VII. *The product of three numbers is the same in whatever manner they are grouped.*

In symbols we have $\boldsymbol{abc = a(bc)}$.

The expression abc without symbols of grouping is understood to mean that the product of a and b is to be multiplied by c.

This axiom states what is called the **associative law of factors** in multiplication.

Principles III and XV of E. C. follow from Axioms VI and VII.

10. Another law for all numbers of algebra is stated as

Axiom VIII. *The product of the sum or difference of two numbers and a given number is equal to the result obtained by multiplying each number separately by the given number and then adding or subtracting the products.*

In symbols we have

$$\boldsymbol{a(b+c) \equiv ab + ac} \text{ and } \boldsymbol{a(b-c) \equiv ab - ac}.$$

Axiom VIII states what is called the **distributive law of multiplication.**

When these identities are read from left to right, they are equivalent to Principle IV, E. C., and when read from right to left (see § 5) they are equivalent to Principles I and II, E. C. The form $a(b \pm c) \equiv ab \pm ac$ is directly applicable to the multiplication of a polynomial by a monomial, and the form $ab \pm ac = a(b \pm c)$, to the addition and subtraction of monomials having a common factor.

11. Definitions. If $ac = b$, the process of finding c when a and b are given is called **division**. This operation is indicated thus: $b \div a = c$, or $\frac{b}{a} = c$, where b is the dividend, a the divisor, and c the quotient. For the case $a = 0$ see §§ 24, 25.

Axiom IX. *If equal numbers are divided by equal numbers (the divisors being different from zero) the quotients are equal numbers.*

Definition. If $a \cdot c = a$, $a \neq 0$,* then the number c is called **unity**, and is written 1. That is, $\frac{a}{a} = 1$. Hence unity is the quotient when dividend and divisor are equal.

By definition of division, the equality $\frac{b}{a} = c$ implies that c is a number such that $ac = b$.

Multiplying both sides of the equality $\frac{b}{a} = c$ by a, we have $a \cdot \frac{b}{a} = ac$, which by hypothesis equals b. Hence *dividing by a number and then multiplying by the same number gives as a result the original number operated upon.*

Axiom IX implies the **uniqueness of division.** That is: *For any two numbers, a and b, $a \neq 0$, there is one and only one number c such that $ac = b$, or $\frac{b}{a} = c$.*

12. Axioms I, IV (in case the subtrahend is not greater than the minuend), V, and IX underlie respectively the processes of addition, subtraction, multiplication, and division, from the very beginning in elementary arithmetic. Axioms II, III, VI, VII, and VIII are also fundamental in arithmetic, where they are usually assumed without formal statement.

E.g. Axiom VIII is used in long multiplication, such as 125×235, where we multiply 125 by 5, by 30, and by 200, and then add the products.

* The symbol $a \neq 0$ stands for the expression *a is not equal to zero.*

13. Negative Numbers. Axiom IV, in case the subtrahend is greater than the minuend, does not hold in arithmetic because of the absence of the negative number. This axiom therefore *brings the negative number into algebra.*

We now proceed to study the laws of operation upon this *enlarged number system.* In the Elementary Course concrete applications were used to show that certain rules of signs hold in operations upon positive and negative numbers. The same rules follow from the axioms just stated.

14. Definitions. If $a+b=0$, then b is said to be the **negative** of a and a the negative of b. If a is a positive number, that is, an ordinary number of arithmetic, then b is called a **negative number.** We denote the negative of a by $-a$. Hence, $a+(-a)=0$. a and $-a$ have the same **absolute value.**

If $a-b$ is positive, then a is said to be *greater than* b. This is written $a>b$. If $a-b$ is negative, then a is said to be *less than* b. This is written $a<b$. If $a-b=0$, then $a=b$. See § 6.

PRINCIPLES OF ADDITION AND SUBTRACTION

15. We now show that

$$\boldsymbol{a+(-b)=a-b}.\qquad \text{See § 48, E. C.}$$

Let $$a+(-b)=x. \tag{1}$$

Adding b to each member of this equation we have, by Axiom I,

$$a+(-b)+b=x+b. \tag{2}$$

But by the associative law,

$$a+(-b)+b=a+[(-b)+b]=a. \tag{3}$$

Hence, $$a=x+b, \text{ or (§ 6), } a-b=x. \tag{4}$$

From (1) and (4), $a+(-b)=a-b.$

That is: *Adding a negative number is equivalent to subtracting this number with its sign changed.*

It follows that either of the symbols, $+(-b)$ and $-b$, may replace the other in any algebraic expression.

16. It is an immediate consequence of § 15 that *a parenthesis preceded by the plus sign may be removed without changing the sign of any term within it.* See § 28, E. C.

It also follows *that an expression may be inclosed in a parenthesis preceded by the plus sign without changing the sign of any of its terms.*

17. To show that $\boldsymbol{a-(-b)=a+b}$. See § 60, E. C.

Let $a-(-b)=x$. (1)

Adding $(-b)$ to both members (Ax. I),

$$a-(-b)+(-b)=x+(-b)=x-b. \quad (2)$$

But $a-(-b)+(-b)=a$. (3)

Hence, $a=x-b$ or $a+b=x$. (4)

From (1) and (4) we have $a-(-b)=a+b$.

That is: *Subtracting a negative number is equivalent to adding this number with its sign changed.*

It follows that either of the symbols $-(-b)$ and $+b$ may replace the other in any algebraic expression.

18. To show that

$$\boldsymbol{a-(b-c+d)=a-b+c-d}.$$ See § 28, E. C.

Let $a-(b-c+d)=x$. (1)

Then $a=x+(b-c+d)=x+b-c+d$. (2)

Adding c and subtracting b and d from each member,

we have $a-b+c-d=x$. (3)

From (1) and (3), $a-(b-c+d)=a-b+c-d$. (4)

That is: *A parenthesis preceded by the minus sign may be removed by changing the sign of each term within it.*

It also follows from equation (4), read from right to left, *that an expression may be inclosed in a parenthesis preceded by a minus sign, if the sign of each term within is changed.*

19. It follows by use of § 18 that

$$\boldsymbol{a-b=-(b-a)}.$$

For $a-b=-b+a=-(b-a)$.

20. It follows further by use of § 18 that

$$-a+(-b)=-(a+b).$$

For $$-a+(-b)=-a-b=-(a+b).$$

21. From the identities

$$a+(-b)\equiv a-b, \qquad \S\ 15,$$

$$a-(-b)\equiv a+b, \qquad \S\ 17,$$

$$a-b\equiv -(b-a), \qquad \S\ 19,$$

$$-a+(-a)\equiv -(a+b), \qquad \S\ 20,$$

it follows that addition and subtraction of positive and negative numbers are reducible to these operations *as found in arithmetic*, where all numbers added and subtracted are positive, and where the subtrahend is never greater than the minuend.

E.g.

$$5+(-8)=5-8=-(8-5)=-3.$$

$$5-(-8)=5+8=13.$$

$$-5-8=-(5+8)=-13.$$

PRINCIPLES OF MULTIPLICATION AND DIVISION

22. *To show that* $a\cdot 0$ *or* $0\cdot a$ *equals* 0 *for all values of* a.

By definition of zero, $a\cdot 0=a(b-b)$.

By Axiom VIII, $a(b-b)=ab-ab$.

By definition of zero, $ab-ab=0$.

Hence, $a\cdot 0=0$.

By Axiom VII, $a\cdot 0=0\cdot a=0$.

It follows that a product is zero if any one of its factors is zero; and if a product is zero, then at least one of its factors must be zero.

23. *To show that* $\frac{0}{a}=0$, *provided* a *is not zero.*

Since by § 22, $0=a\cdot 0$, we have by the definition of division $\frac{0}{a}=0$.

24. *To show that* $\frac{0}{0}$ *represents any number whatever. That is,* $\frac{0}{0}=k$, *for all values of* k.

Since by § 22, $0 = 0 \cdot k$, we have by the definition of division $\frac{0}{0} = k$ for all values of k. Hence, $\frac{0}{0}$ does not represent any *definite* number.

25. *To show that there is no number* $\boldsymbol{k}$ *such that* $\frac{\boldsymbol{a}}{\boldsymbol{0}} = \boldsymbol{k}$, *provided* $\boldsymbol{a}$ *is not zero.*

If $\frac{a}{0} = k$, then by definition of division, $k \cdot 0 = a$. But by § 22, $k \cdot 0 = 0$ for all values of k. Hence, if a is not zero, k is *impossible.*

From §§ 24, 25, it follows that *division by zero is to be ruled out in all cases* unless special interpretation is given to the results thus obtained.

26. *To show that* $\boldsymbol{a(-b) = -ab}$. See § 63, E. C.

Let $$a(-b) = x. \qquad (1)$$

Adding ab to both members,

$$a(-b) + ab = x + ab. \qquad (2)$$

By Axiom VIII, $$a[(-b) + b] = x + ab, \qquad (3)$$

or, $$a \cdot 0 = 0 = x + ab. \qquad (4)$$

Hence (§ 14), $$x = -ab. \qquad (5)$$

From (1) and (5), $$a(-b) = -ab.$$

That is, *the product of a positive and a negative number is negative.*

27. *To show that* $\boldsymbol{(-a)(-b) = ab}$. See § 63, E. C.

Let $$(-a)(-b) = x. \qquad (1)$$

Adding $(-a)b$ to each member,

$$(-a)(-b) + (-a)b = x + (-a)b = x - ab. \qquad (2)$$

By Axiom VIII, $$(-a)[(-b) + b] = x - ab, \qquad (3)$$

or, $$0 = x - ab. \qquad (4)$$

Hence (§ 14), $$ab = x. \qquad (5)$$

From (1) and (5), $$(-a)(-b) = ab.$$

That is, *the product of two negative numbers is positive.*

28. *To show that if the signs of the dividend and divisor are alike, the quotient is positive; and if unlike, the quotient is negative.* See § 67, E. C.

If $a = bc$, then by the law of signs in multiplication, $-a = (-b)c$, $-a = b(-c)$, and $a = (-b)(-c)$. Hence, by definition of division, we have respectively:

$$\frac{a}{b} = c,\ \frac{-a}{-b} = c,\ \frac{-a}{b} = -c, \text{ and } \frac{a}{-b} = -c.$$

29. *To show that* $\frac{a}{c} \cdot b = \frac{ab}{c}$.

Let $$x = \frac{a}{c} \cdot b. \tag{1}$$

Then $$cx = c \cdot \frac{a}{c} \cdot b = ab. \tag{2}$$

Dividing by c, $$x = \frac{ab}{c}. \tag{3}$$

From (1) and (3), $$\frac{a}{c} \cdot b = \frac{ab}{c}. \tag{4}$$

30. *To show that* $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

By § 29, $$\frac{a+b}{c} = \frac{1 \cdot (a+b)}{c} = \frac{1}{c}(a+b). \tag{1}$$

By Axiom VIII, $$\frac{1}{c}(a+b) = \frac{1}{c} \cdot a + \frac{1}{c} \cdot b = \frac{a}{c} + \frac{b}{c}. \tag{2}$$

Hence, $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}. \tag{3}$$

31. *To show that* $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$.

By § 29 $$\frac{a-b}{c} = \frac{1 \cdot (a-b)}{c} = \frac{1}{c}(a-b). \tag{1}$$

By Axiom VIII, $$\frac{1}{c}(a-b) = \frac{1}{c} \cdot a - \frac{1}{c} \cdot b = \frac{a}{c} - \frac{b}{c}. \tag{2}$$

Hence, $$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}. \tag{3}$$

CHAPTER II

FUNDAMENTAL OPERATIONS

32. The operations of addition, subtraction, multiplication, division, and finding powers and roots are called **algebraic operations.**

33. An **algebraic expression** is any combination of number symbols (Arabic figures or letters or both) by means of indicated algebraic operations.

E.g. 24, $3+7$, $9(b+c)$, $\frac{m+n}{k}$, $x^2+\sqrt{y}$, are algebraic expressions.

34. Any number symbol upon which an algebraic operation is to be performed is called an **operand.**

All the algebraic operations have been used in the Elementary Course. They are now to be considered in connection with the fundamental laws developed in the preceding chapter, and then applied to more complicated expressions. The finding of powers and roots will be extended to higher cases.

35. One of the two equal factors of an expression is called the **square root** of the expression; one of the three equal factors is called its **cube root**; one of the four equal factors, its **fourth root**, etc. A root is indicated by the **radical sign** and a number, called the **index** of the root, which is written within the sign. In the case of the square root, the index is omitted.

E.g. $\sqrt{4}$ is read *the square root of* 4; $\sqrt[3]{8}$ is read *the cube root of* 8; $\sqrt[4]{64}$ is read *the fourth root of* 64, etc.

36. A root which can be expressed in the *form* of an integer, or as the quotient of two integers, is said to be **rational,** while one which cannot be so expressed is **irrational.**

E.g. $\sqrt[3]{8}=2$, $\sqrt{a^2+2ab+b^2}=a+b$, and $\sqrt{\frac{9}{4}}=\frac{3}{2}$ are rational roots, while $\sqrt[3]{4}$ and $\sqrt{a^2+ab+b^2}$ are irrational roots.

An algebraic expression which involves a letter in an irrational root is said to be **irrational with respect to that letter;** otherwise the expression is rational with respect to the letter.

E.g. $a + b\sqrt{c}$ is rational with respect to a and b, and irrational with respect to c.

37. An expression is **fractional** with respect to a given letter if after reducing its fractions to their lowest terms the letter is still contained in a denominator.

E.g. $\dfrac{a}{c+d} + b$ is fractional with respect to c and d, but not with respect to a and b.

38. **Order of Algebraic Operations.** In a series of indicated operations where no parentheses or other symbols of aggregation occur, it is an established usage that the operations of finding powers and roots are to be performed first, then the operations of multiplication and division, and finally the operations of addition and subtraction.

$$E.g.\ 2 + 3 \cdot 4 + 5 \cdot \sqrt[3]{8} - 4^2 \div 8 = 2 + 3 \cdot 4 + 5 \cdot 2 - 16 \div 8$$
$$= 2 + 12 + 10 - 2 = 22.$$

In cases where it is necessary to distinguish whether multiplication or division is to be performed first, parentheses are used.

E.g. In $6 \div 3 \times 2$, if the division comes first, it is written $(6 \div 3) \times 2 = 4$, and if the multiplication come first, it is written $6 \div (3 \times 2) = 1$.

ADDITION AND SUBTRACTION OF MONOMIALS

39. In accordance with § 10, the sum (or difference) of terms which are *similar with respect to a common factor* (§ 78, E. C.) is equal to the product of this common factor and the sum (or difference) of its coefficients.

Ex. 1. $8ax^2 + 9ax^2 - 3ax^2 = (8 + 9 - 3)ax^2 = 14ax^2.$

Ex. 2. $a\sqrt{x^2+y^2} + b\sqrt{x^2+y^2} = (a+b)\sqrt{x^2+y^2}.$

Ex. 3. $$\frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + \frac{x(x-1)}{1 \cdot 2} = \left(\frac{x-2}{3} + 1\right)\frac{x(x-1)}{1 \cdot 2}$$
$$= \frac{x+1}{3} \cdot \frac{x(x-1)}{1 \cdot 2} = \frac{(x+1)x(x-1)}{1 \cdot 2 \cdot 3}.$$

EXERCISES

Perform the following indicated operations:

1. $5x^4b^2 - 3x^4b^2 - 4x^4b^2 + 7x^4b^2$.

2. $3\sqrt{x^2-4} - 2\sqrt{x^2-4} + 2\sqrt{x^2-4} - 4\sqrt{x^2-4}$.

3. $ab^5c^4 - db^5c^4 + eb^5c^4 + fb^5c^4$.

4. $a^6x^4 + 5a^5x^4 - 5a^5x^5 - 3a^5x^4$.

5. $7x^3y^5 + 5x^4y^4 - 9x^4y^5 + 5x^3y^4$.

6. $2a^n + a^{n-1} + a^{n+1} = a^{n-1}(2a + 1 + a^2) = a^{n-1}(1+a)^2$.

7. $n(n-1)(n-2)(n-3)(n-4) + n(n-1)(n-2)(n-3)$.

$n(n-1)(n-2)(n-3)$ is the common factor and $n-4$ and 1 are the coefficients to be added.

8. $n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$
$+ n(n-1)(n-2)(n-3)(n-4)(n-5)$.

9. $n(n-1)(n-2)(n-3) + (n-1)(n-2)(n-3)$.

10. $n(n-1)(n-2)(n-3)(n-4) + (n-1)(n-2)(n-3)$.

11. $(a-4)(b+3) + (a-1)(b-2) + (a+3)(b+3)$.

First add $(a-4)(b+3)$ and $(a+3)(b+3)$.

12. $(x+2y)(x-2y) + (x-3y)(x-2y) - (2x-y)(x-y)$.

13. $(5a-3b)(a-b)(a+b) + (2b-4a)(a-b)(a+b)$
$+ (a-b)^2(2a-b)$.

14. $(7x^2+3y^2)(5x-y)(x+y) + (7x^2+3y^2)(x+y)(2y-4x)$
$+ (7x^2-3y^2)(x+y)^2$.

15. $2^3 \cdot 3^2 \cdot 5 + 2^4 \cdot 3 \cdot 5$.

The common factor is $2^3 \cdot 3 \cdot 5$. Hence the sum is

$$2^3 \cdot 3 \cdot 5(3+2) = 2^3 \cdot 3 \cdot 5^2.$$

16. $2 \cdot 3^4 \cdot 7 + 2^2 \cdot 3^3 \cdot 7^2 - 2^4 \cdot 3^3 \cdot 7$.

17. $3^4 \cdot 5^7 \cdot 13 + 3^5 \cdot 5^7 \cdot 13^2$.

18. $5^4 \cdot 7^3 \cdot 11 + 5^3 \cdot 7^2 \cdot 11 - 2^3 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 11.$

19. $3^{22} \cdot 7^{18} \cdot 13^{15} + 3^{21} \cdot 7^{17} \cdot 13^{15} + 3^{24} \cdot 7^{17} \cdot 13^{15}.$

20. $1 \cdot 2 \cdot 3 \cdots n \;+ 1 \cdot 2 \cdot 3 \cdots n(n+1).$

The dots mean that the factors are to run on in the manner indicated up to the number n. The common factor in this case is $1 \cdot 2 \cdot 3 \cdots n$, and the coefficients to be added are 1 and $n + 1$. Hence the sum is $1 \cdot 2 \cdot 3 \cdots n(n + 2)$.

21. $1 \cdot 2 \cdot 3 \cdots n \;+ 1 \cdot 2 \cdot 3 \cdots n(n+1)$
$$+ 1 \cdot 2 \cdot 3 \cdots n(n+1)(n+2).$$

22. $1 \cdot 2 \cdot 3 \cdots n \;+ 3 \cdot 4 \cdot 5 \cdots n \;+ 5 \cdot 6 \cdot 7 \cdots n.$

23. $n(n-1) \cdots (n-6) \;+ n(n-1) \cdots (n-6)(n-7).$

24. $n(n-1) \cdots (n-r) \;+ n(n-1) \cdots (n-r)(n-r-1).$

25. $na^nb + a^nb.$

26. $\dfrac{n(n-1)}{1 \cdot 2} a^{n-1}b^2 + na^{n-1}b^2.$

27. $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-2}b^3 + \dfrac{n(n-1)}{1 \cdot 2} a^{n-2}b^3.$

The common factor is $\dfrac{n(n-1)}{1 \cdot 2} a^{n-2}b^3$ and the coefficients to be added are $\dfrac{n-2}{3}$ and 1.

28. $\dfrac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} a^{n-3}b^4 + \dfrac{n(n-1)(n-2)}{2 \cdot 3} a^{n-3}b^4.$

29. $\dfrac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4 \cdot 5} a^{n-4}b^5$
$$+ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} a^{n-4}b^5.$$

30. $\dfrac{n(n-1) \cdots (n-r+1)(n-r)}{2 \cdot 3 \cdots r(r+1)} a^{n-r}b^{r+1}$
$$+ \frac{n(n-1) \cdots (n-r+1)}{2 \cdot 3 \cdots r} a^{n-r}b^{r+1}.$$

ADDITION AND SUBTRACTION OF POLYNOMIALS

40. The **addition** of polynomials is illustrated by the following example.

Add $2a + 3b - 4c$ and $3a - 2b + 5c$.

The sum may be written thus:

$$(2a + 3b - 4c) + (3a - 2b + 5c).$$

By the associative law, § 5, and by § 16, we have,

$$2a + 3b - 4c + 3a - 2b + 5c.$$

By the commutative law, § 4, and by § 15, this becomes,

$$2a + 3a + 3b - 2b - 4c + 5c.$$

Again by the associative law, combining similar terms, we have,

$$5a + b + c.$$

From this example it is evident that several polynomials may be added by combining similar terms and then indicating the sum of these results.

For this purpose the polynomials are conveniently arranged so that similar terms shall be in the same column. Thus, in the above example,

$$\begin{array}{r} 2a + 3b - 4c \\ 3a - 2b + 5c \\ \hline 5a + b + c \end{array}$$

41. For **subtraction** the terms of the polynomials are arranged as for addition. The subtraction itself is then performed as in the case of monomials. See §§ 17–19.

Example. Subtract $4x - 2y + 6z$ from $3x + 6y - 3z$.

$$\begin{array}{r} 3x + 6y - 3z \\ 4x - 2y + 6z \\ \hline -x + 8y - 9z \end{array}$$

The steps are:

$$3x - 4x = -x; \quad 6y - (-2y) = 8y; \quad -3z - (+6z) = -9z.$$

EXERCISES

1. Add $8x^3 - 11x - 7x^2$, $2x - 6x^2 + 10$, $-5 + 4x^3 + 9x$, and $13x^2 - 5 - 12x^3$.

2. Add $5a^3 - 2a - 12 - 10a^2$, $14 - 7a + a^2 - 9a^3$, $3a^2 - 13a^3 + 4 - 11a$, and $3 - 7a + 10a^2 + 4a^3$.

3. From the sum of $9m^3 - 3m^2 + 4m - 7$ and $3m^2 - 4m^3 + 2m + 8$ subtract $4m^3 - 2m^2 - 4 + 8m$.

4. From the sum of $x^4 - ax^3 - a^2x^2 - a^3x + 2a^4$ and $3ax^3 + 7a^2x^2 - 5a^3x + 2a^4$ subtract $3x^4 + ax^3 - 3a^2x^2 + a^3x - a^4$.

5. Add $37a - 4b - 17c + 15d - 6f - 8h$ and $3c - 31a + 9b - 5d - h - 4f$.

6. Add $11q - 10p - 8n + 3m$, $24m - 17q + 15p - 13n$, $9n - 6m - 4q - 7p - 5n$, and $8q - 4p - 12m + 18n$.

7. From the sum of $13a - 15b - 7c - 11d$ and $7a - 6b + 8c + 3d$ subtract the sum of $6d - 5b - 7c + 2a$ and $5c - 10d - 28b + 17a$.

8. Add $2^3 \cdot 3^4 x^3 - 2^5 \cdot 3^2 x^2 + 2^2 \cdot 3^3 \cdot 7x + 2^2 \cdot 3^2 \cdot 5$, $2^2 \cdot 3^3 x^3 - 2^4 \cdot 3^2 \cdot 7x + 2^4 \cdot 3^3 x^2 - 2^2 \cdot 3^2 \cdot 5^2$, and $2^3 \cdot 3^3 x^3 - 2^3 \cdot 3^3 \cdot 5 + 2^3 \cdot 3^3 x - 2^4 \cdot 3^4 x^2$.

9. Add $(a + b - c)m + (a - b + c)n + (a - b - c)k$,
$(2a - 3b + c)m + (b - 3a + c)n + (4c + 2b + a)k$,
and $(b - 2c)m + (2a - 2c + b)n + (2b - 2a + c)k$.

10. From the sum of $ax^3 - bx^2 + cx - d$ and $bx^3 + ax^2 - dx + c$ subtract $(a - b)x^3 + (c - a)x^2 - (b + d)x - d + c$.

11. From $(m - n)(m - n)x^3 + (n - m)^2x^2 - (n + m)x + 8$ subtract the sum of $n(m - n)x^3 - 4(n - m)^2x^2 + (n + m)x - 31$ and $2(n - m)^2x^2 - m(m - n)x^3 - 2(n + m)x + 25$.

12. Add $a^n + 2a^{n+1} + a^{n+2}$ and $2a^n - 4a^{n+1} + 5a^{n+2}$ and om this sum subtract $7a^{n+1} - 8a^n + a^{n+2}$.

REMOVAL OF PARENTHESES

42. By the principles of §§ 15–18, a parenthesis inclosing a polynomial may be removed with or without the change of sign of each term included, according as the sign $-$ or $+$ precedes the parenthesis.

In case an expression contains signs of aggregation, one within another, these may be removed *one at a time*, beginning with the *innermost*, as in the following example:

$$
\begin{aligned}
&a - \{b + c - [d - e + f - (g - h)]\} \\
&= a - \{b + c - [d - e + f - g + h]\} \\
&= a - \{b + c - d + e - f + g - h\} \\
&= a - b - c + d - e + f - g + h.
\end{aligned}
$$

Such involved signs of aggregation may also be removed *all at once*, beginning with *the outermost*, by observing the *number of minus signs* which affect each term, and calling the sign of any term $+$ if this number is *even*, $-$ if this number is *odd*.

Thus, in the above example, b and c are each affected by *one* minus sign, namely, the one preceding the brace. Hence we write, $a - b - c$.

d and f are each affected by *two* minus signs, namely the one before the brace and the one before the bracket, while e is affected by these two, and also by the one preceding it. Hence we write, $d - e + f$.

g is affected by the minus signs before the bracket, the brace, and the parenthesis, an *odd* number, while h is affected by these and also by the one preceding it, an *even* number. Hence we write $-g + h$.

By counting in this manner as we proceed from left to right, we give the final form at once, $a - b - c + d - e + f - g + h$.

EXERCISES

In removing the signs of aggregation in the following, either process just explained may be used. The second method is shorter and should be easily followed after a little practice.

1. $7 - \{-4 - (4 - [-7]) - (5 - [4 - 5] + 2)\}$.

2. $-[-(7-\{-4+9\}-13)-(12-3+[-7+2])]$.

3. $6-(-3-[-5+4]+\{7-3-(7-19)\}+8)$.

4. $5+[-(-\{-5-3+11\}-15)-3]+8$.

5. $4x-[3x-y-\{3x-y-(x-\overline{y-x})+x\}-3y]$.

The vinculum above $y-x$ has the same effect as a parenthesis, *i.e.* $-\overline{y-x}=-(y-x)$.

6. $3x^2-2y^2-(4x^2-\{3x^2-(y^2-2x^2)-3y^2\}-y^2+4x^2)$.

7. $7a-\{3a-[-2a-\overline{a+3}+a]-\overline{2a-5}\}$.

8. $l-(-2m-n-\{l-m\})-(5l-2n-[-3m+n])$.

9. $2d-[3d+\{2d-(e-5d)\}-(d+3e)]$.

10. $4y-(-2y-[-3y-\{-y-\overline{y-1}\}+2y])$.

11. $3x-[8x-(x-3)-\{-2x+6-\overline{8x-1}\}]$.

12. $x-(x-\{-4x-[5x-\overline{2x-5}]-[-x-\overline{x-3}]\})$.

13. $3x-\{y-[3y+2z]-(4x-[2y-3z]-\overline{3y-2z})+4x\}$.

14. $x-(-x-\{-3x-[x-\overline{2x+5}]-4\}-[2x-\overline{x-3}])$.

MULTIPLICATION OF MONOMIALS

43. In the elementary course we saw that $2^k\cdot 2^n=2^{k+n}$. More generally, if b is any number and k and n any positive integers, we have

$$b^k\cdot b^n=b^{k+n}.$$

For by the definition of a positive integral exponent,

$$b = b\cdot b\cdot b \cdots \text{ to } k \text{ factors},$$

and $$b^n = b\cdot b\cdot b \cdots \text{ to } n \text{ factors}.$$

Then, $$b^k\cdot b^n = (b\cdot b \cdots \text{ to } k \text{ factors})(b\cdot b \cdots \text{ to } n \text{ factors})$$
$$= b\cdot b\cdot b \cdots \text{ to } k+n \text{ factors}.$$

Hence, $$b^k\cdot b^n = b^{k+n}.$$

That is: *The product of two powers of the same base is a power of that base whose exponent is the sum of the exponents of the common base.*

44. In finding the product of two monomials, the factors may be *arranged* and *associated* in any manner, according to §§ 8, 9.

E.g. $(3ab^2) \times (5a^2b^3) = 3ab^2 \cdot 5a^2b^3$ § 9

$= 3 \cdot 5 \cdot a \cdot a^2 \cdot b^2 \cdot b^3$ § 8

$= (3 \cdot 5)(a \cdot a^2)(b^2 \cdot b^3)$ § 9

$= 15a^3b^5$ § 43

The factors in the product are arranged so as to associate those consisting of Arabic figures and also those which are powers of the same base. This arrangement and association of the factors is equivalent to multiplying either monomial by the factors of the other in succession. See § 129, E. C.

45. It is readily seen that a product is negative when it contains an *odd* number of *negative* factors; otherwise it is positive.

For by the commutative and associative laws of factors the negative factors may be grouped in *pairs*, each pair giving a *positive* product. If the number of negative factors is odd, there will be just one remaining, which makes the final product negative.

EXERCISES

Find the products of the following:

1. $2^3 \cdot 3^4 \cdot 4^7,\ 2^7 \cdot 3^2 \cdot 4^2$.
2. $3 \cdot 2^4 \cdot 5^2,\ 5 \cdot 2^2 \cdot 5,\ 7 \cdot 2^3 \cdot 5^3$.
3. $2x^2y^3,\ 5x^3y^2,\ 2x^4y$.
4. $5xy,\ 2x^3y,\ 4xy^5,\ x^2y^2$.
5. $3a^5bc,\ ab^2c,\ a^2bc^4,\ 4ab^5c$.
6. $x^n,\ x^{n-1},\ x^{n+1},\ 2x^n$.
7. $x^{m+n-1},\ x^{m-n+1},\ x^{2m}$.
8. $a^x,\ a^{3x-y},\ a^{y-3x}$.
9. $a^nb^m,\ a^{2n}b^{3m},\ a^{1-3n}b^{2-4m}$.
10. $4ab^m,\ 2a^3b^n,\ 3a^6b^{2-m-n}$.
11. $2x^my^{m+n},\ 3x^{m-1}y^{2n-m+2}$.
12. $a^{d-2c+2}b^{m-3n},\ a^{2c-d-1}b^{2-m+3n}$.
13. $3x^{a+3b},\ 2x^{a-2b}y^{c-3},\ 2x^{4-2a-b}y^{2c+3}$.
14. $a^{2x-3}b^{y+1},\ a^{x+3}b^{y-1},\ 3a^3b^2$.
15. $3^{4a-2-2b} \cdot 2^{n+3-m},\ 3^{5-4a+2b} \cdot 2^{m+2-n}$.
16. $x^{3x+1+y},\ x^{y-2x-1}y^{2x},\ y^{1-x}$.
17. $7 \cdot 2^{3a-4},\ 3 \cdot 2^{5-2a},\ 5 \cdot 2^{3-a}$.
18. $2^{7x-1-4y},\ 3 \cdot 2^{1-5x-4y},\ 3^2 \cdot 2^{2-2x}$.
19. $3^{2-5m+3n} \cdot 2^{4a-3b},\ 3^{2-3n+6m} \cdot 2^{5+3b+5a}$.
20. $(1+a)^{7-3b+a} \cdot (1-a)^{2+a-b},\ (1-a)^{b-a-1} \cdot (1+a)^{3b-a-6}$.

DIVISION OF MONOMIALS

46. In the Elementary Course we have seen that $x^6 \div x^4 = x^{6-4} = x^2$, etc. In general, if a is any number and m and k are any positive integers, of which m is the greater, then

$$a^m \div a^k = a^{m-k}.$$

For, since k and $m-k$ are both positive integers, we have, by § 43, $a^k a^{m-k} = a^{k+m-k} = a^m$. That is, a^{m-k} is the number which multiplied by a^k gives a product a^m, and hence by the definition of division,

$$a^m \div a^k = a^{m-k}.$$

Hence: *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

Under the proper interpretation of negative numbers used as exponents this principle also holds when $m < k$. This is considered in detail in § 177. We remark here that in case $m = k$, the dividend and the divisor are equal and the quotient is unity. Hence $a^m \div a^m = a^{m-m} = a^0 = 1$. See § 11.

47. We have seen in the earlier work that $\frac{4x}{6y} = \frac{2x}{3y}$, etc. In general, if a, b, and k are any number expressions:

$$\frac{ak}{bk} = \frac{a}{b}.$$

For, by definition of division, $\quad a = \frac{a}{b} \cdot b. \quad$ (1)

Multiplying both sides of (1) by k, $\quad ak = \frac{a}{b} \cdot bk. \quad$ (2)

Dividing both sides of (2) by bk, $\quad \frac{ak}{bk} = \frac{a}{b}. \quad$ (3)

Hence: *In dividing one algebraic expression by another, all factors common to dividend and divisor may be removed or canceled.*

EXERCISES

Divide:

1. $4 \cdot 2^4 \cdot 3^7 \cdot 5^2$ by $3 \cdot 2^3 \cdot 3^4 \cdot 5$.
2. $5 \cdot 3^7 \cdot 7^4 \cdot 13^5$ by $2 \cdot 3^5 \cdot 7^2 \cdot 13^2$.
3. $3x^7y^2z$ by $2x^3yz$.
4. $5a^5b^7c^8$ by $5a^4b^7c^4d^2$.
5. $x^{2n}y^mz^{3m}$ by $x^ny^mz^m$.
6. $a^{3n-5}y^{2n+3}$ by $a^{n+6}y^{2n+1}$.
7. $a^{c+3d+2}b^{d-2c+6}$ by a^{c+2d-4}.
8. $3^{a+2b-7} \cdot 5^{3b-2a+4}$ by $3^{b+a-8} \cdot 5^{2b-2a+3}$.
9. $a^{3+2m-3n}b^5c^{7-n}$ by $a^{2+m-4n}b^4c^{7-n}$.
10. $x^{4a-2b+1}y^{c-a+b}z^{3a+2b+c}$ by $z^{2b-c+3a}y^{a-c+b}x^{3a+b-2c}$.
11. $2^{3a-4+7b} \cdot 3^{3b-4c+6}$ by $2^{2a-5-7b} \cdot 3^{2b-6c+7}$.
12. $(x-2)^{3m+1-3n} \cdot (x+2)^{2m+2-3n}$ by $(x+2)^{1+2m-2n} \cdot (x-2)^{1-3n+2m}$.
13. $(x-y)^{5b-3c-1} \cdot (x+y)^{7c-2b+2}$ by $(x-y)^{-2-3c+5b} \cdot (x+y)^{-3-2b+7c}$.
14. $(a^2-b^2)^{3+4k+7b} \cdot (a^2-b^2)^{1-3k-5b}$ by $(a^2-b^2)^{4+k} \cdot (a^2-b^2)^{-2+2b}$.

MULTIPLICATION OF POLYNOMIALS

48. In § 86, E.C., we saw that *the product of two polynomials is equal to the sum of the products obtained by multiplying each term of one polynomial by every term of the other.*

This follows from the distributive law of multiplication, § 10. For by this law,

$$\begin{aligned}(m+n+k)(a+b+c) &= m(a+b+c)\\ &+ n(a+b+c)\\ &+ k(a+b+c).\end{aligned}$$

Applying the same law to each part, we have the product,

$$ma + mb + mc + na + nb + nc + ka + kb + kc.$$

EXERCISES

Find the following indicated products:

1. $(a+b)(a+b)$, i.e. $(a+b)^2$; also $(a-b)^2$.
2. $(a+b)(a+b)(a+b)$, i.e. $(a+b)^3$; also $(a-b)^3$.
3. $(a+b)(a+b)(a+b)(a+b)$, i.e. $(a+b)^4$; also $(a-b)^4$.
4. $(a^2+2ab+b^2)(a^2+2ab+b^2)(a+b)$.
5. $(a^2-2ab+b^2)(a^2-2ab+b^2)(a-b)$.
6. $(a^2+2ab+b^2)^3$; also $(a+b)^6$.
7. $(a^3+3a^2b+3ab^2+b^3)^2$.
8. $(a^3-3a^2b+3ab^2-b^3)^2$.
9. $(a-b)(a^2+ab+b^2)$.
10. $(a+b)(a^2-ab+b^2)$.
11. $(a^2+2ab+b^2)(a^4+4a^3b+6a^2b^2+4ab^3+b^4)$.
12. $(a^2-2ab+b^2)(a^4-4a^3b+6a^2b^2-4ab^3+b^4)$.
13. $(a-b)(a^3+a^2b+ab^2+b^3)$.
14. $(a+b)(a^3-a^2b+ab^2-b^3)$.
15. $(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.
16. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$.
17. $(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)$.
18. $(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5)$.
19. $(1-r)(a+ar+ar^2+ar^3)$.
20. $(1-r)(a+ar+ar^2+ar^3+ar^4+ar^5)$.
21. $(a+b+c)^2$.
22. $(a+b-c)^2$.
23. $(a-b-c)^2$.
24. From Exs. 21–23 deduce a rule for squaring a trinomial.
25. $(x+y+z+v)^2$.
26. $(x-y+z-v)^2$.
27. From Exs. 25, 26 deduce a rule for squaring a polynomial.
28. $(a+b+c)(a+b-c)(a-b+c)(b-a+c)$.
29. $[(x+y)(x+y)(x-y)(x-y)]^2$.
30. $[(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8-b^8)]^2$.

31. $(4x^2-6xy+9y^2)(2x+3y)(4x^2+6xy+9y^2)(2x-3y)$.

32. Collect in a table the following products:

$(a+b)^2$, $(a-b)^2$, $(a+b)^3$, $(a-b)^3$, $(a+b)^4$.
$(a-b)^4$, $(a+b)^5$, $(a-b)^5$, $(a+b)^6$, $(a-b)^6$.

33. From the above table answer the following questions:

(*a*) How many terms in each product, compared with the exponent of the binomial?

(*b*) Tell how the signs occur in the various cases.

(*c*) How do the exponents of a proceed? of b?

(*d*) Make a table of the coefficients alone and memorize this.

E.g. For $(a+b)^5$, they are 1, 5, 10, 10, 5, 1.

34. Make use of the rules in Exs. 24, 27, 33 to write the following products: (*a*) $(2x-3y+4z)^2$, (*b*) $(\frac{1}{2}m^2-\frac{1}{4}n^3-3r)^2$, (*c*) $(4ax-2ay+3m-n)^2$.

(*d*) $(2x+3y)^3$.
(*e*) $(3a-b^2)^4$.
(*f*) $(\frac{1}{2}ax-\frac{2}{3}by)^3$.
(*g*) $(2m-n)^5$.
(*h*) $\left(\frac{a}{3}-\frac{b}{4}\right)^4$.
(*i*) $\left(2x-\frac{y}{2}\right)^5$.
(*j*) $\left(\frac{a}{2}-x^2\right)^3$.
(*k*) $(9x-2y)^2$.
(*l*) $(1-2x)^4$.
(*m*) $(1+3x)^5$.

DIVISION OF POLYNOMIALS

49. According to the distributive law of division, § 30, a *polynomial is divided by a monomial* by dividing each term separately by the monomial. See § 25, E. C.

E.g. $\dfrac{ab+ac-ad}{a}=\dfrac{ab}{a}+\dfrac{ac}{a}-\dfrac{ad}{a}=b+c-d.$

A polynomial is divided by a polynomial by separating the dividend into polynomials, each of which is the product of the divisor and a monomial. Each of these monomial factors is a part of the quotient, their sum constituting the whole quotient. The parts of the dividend are found one by one as the work proceeds. See §§ 161–163, E.C. This is best shown by an example.

$$\begin{array}{lrl}
\text{Dividend,} & a^4 + \quad a^3 - 4a^2 + 5a - 3 & \underline{|\, a^2 + 2a - 3,\ \text{Divisor.}} \\
\text{1st part of dividend:} & \underline{a^4 + 2a^3 - 3a^2 \qquad\qquad} & |\, a^2 - a + 1, \quad \text{Quotient.} \\
 & -a^3 - a^2 \ + 5a - 3 & \\
\text{2d part of dividend:} & \underline{-a^3 - 2a^2 + 3a \quad} & \\
 & a^2 + 2a - 3 & \\
\text{3d part of dividend:} & \underline{a^2 + 2a - 3} & \\
 & 0 &
\end{array}$$

The three parts of the dividend are the products of the divisor and the three terms of the quotient. If after the successive subtraction of these parts of the dividend the remainder is zero, the division is exact. In case the division is not exact, there is a final remainder such that

$$\textit{Dividend} = \textit{Quotient} \times \textit{divisor} + \textit{Remainder}.$$

In symbols we have $\mathbf{D} = \mathbf{Q} \cdot \mathbf{d} + \mathbf{R}$.

EXERCISES

Divide:

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
2. $x^8 + x^4y^4 + y^8$ by $x^4 - x^2y^2 + y^4$.
3. $x^5 - y^5$ by $x - y$.
4. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
5. $x^9 + y^9$ by $x^2 - xy + y^2$.
6. $x^8 - y^8$ by $x^2 - y^2$.
7. $a^6 + b^6$ by $a^2 + b^2$.
8. $x^{6a} - y^{6b}$ by $x^{2a} - y^{2b}$.
9. $a^{10} - a^5b^5 + b^{10}$ by $a^2 - ab + b^2$.
10. $2x^4 - 3x^3b + 6x^2b^2 - xb^3 + 6b^4$ by $x^2 - 2xb + 3b^2$.
11. $2x^6 - 5x^5 + 6x^4 - 6x^3 + 6x^2 - 4x + 1$ by $x^4 - x^3 + x^2 - x + 1$.
12. $26a^3b^3 + a^6 + 6b^6 - 5a^5b - 17ab^5 - 2a^4b^2 - a^2b^4$ by $a^2 - 3b^2 - 2ab$.
13. $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $x^2 - 3x + 2$.
14. $4b^2 + 4ab + a^2 - 12bc - 6ac + 9c^2$ by $2b + a - 3c$.
15. $x^4 + 4xy^3 - 4xyz + 3y^4 + 2y^2z - z^2$ by $x^2 - 2xy + 3y^2 - z$.
16. $a^2b^2c + 3a^2b^3 - 3abc^3 - a^2c^3 + b^5 - 4b^3c^2 + 3ab^3c + 3bc^4 - 3a^2bc^2$ by $b^2 - c^2$.

CHAPTER III

INTEGRAL EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

50. When in an algebraic expression a letter is replaced by another number symbol, this is called a **substitution on that letter.**

E.g. In the expression, $2a + 5$, if a is replaced by 3, giving $2 \cdot 3 + 5$, this is a substitution on the letter a.

51. An equality containing a single letter is said to be satisfied by any substitution on that letter which reduces both members of the equality to the same number.

E.g. $4x + 8 = 24$ is satisfied by $x = 4$, since $4 \cdot 4 + 8 = 24$.

We notice, however, that the substitution *must not reduce the denominator of any fraction to zero.*

Thus $x = 2$ does *not* satisfy $\frac{x^2 - 4}{x - 2} = 8$ although it reduces the left member of the equation to $\frac{0}{0}$, which by § 24 equals 8 or *any other number whatever.*

On the other hand, $x = 6$ satisfies this equation, since

$$\frac{6^2 - 4}{6 - 2} = \frac{32}{4} = 8.$$

52. An equality in two or more letters is **satisfied** by any simultaneous substitutions on these letters which reduce both members to the same number.

E.g. $6a + 3b = 15$ is satisfied by $a = 2, b = 1$; $a = \frac{3}{2}, b = 2$; $a = 1, b = 3$, etc.

$\frac{x^2 - y^2}{x^2 + 2xy + y^2} = \frac{1}{2}$ is satisfied by $x = 3, y = 1$, but is *not* satisfied by any values of x and y such that $x = -y$, since these reduce the denominator (and also the numerator) to zero. See § 24.

53. An equality is said to be an **identity in all its letters,** or simply an **identity,** if it is satisfied by *every possible substitution* on these letters, not counting those which make any denominator zero.

If an equality is an identity, both members will be reduced to the same expression when all indicated operations are performed as far as possible.

The members of an identity are called **identical expressions.**

Thus in the identity $(a + b)^2 \equiv a^2 + 2\,ab + b^2$, performing the indicated operation in the first member reduces it to the same form as the second.

54. An equality which is not an identity is called an **equation of condition** or simply an **equation.**

The members of an equation *cannot* be reduced to the same expression by performing the indicated operations.

E.g. $(x - 2)(x - 3) = 0$ cannot be so reduced. This is an equation which is satisfied by $x = 2$ and $x = 3$. See § 22.

55. In an *equation* containing several letters any one or more of them may be regarded as **unknown,** the remaining ones being considered **known.** Such an equation is said to be satisfied by any substitution on the *unknown* letters which reduces it to an *identity in the remaining letters.*

E.g. $x^2 - t^2 = sx + st$ is an equation in s, x, or t, or in any pair of these letters, or in all three of them.

As an equation in x it is satisfied by $x = s + t$, since this substitution reduces it to the identity in s and t,

$$s^2 + 2\,st \equiv s^2 + 2\,st.$$

As an equation in s it is satisfied by $s = x - t$, since this substitution reduces it to the identity in x and t,

$$x^2 - t^2 \equiv x^2 - t^2.$$

Any number expression which satisfies an equation in one unknown is called a **root of the equation.**

E.g. $s + t$ is a root of the equation $x^2 - t^2 = sx + st$, when x is the unknown, and $x - t$ is a root when s is the unknown.

56. An equation is **rational in a given letter** if every term in the equation is rational with respect to that letter.

An equation is **integral** in a given letter if every term is rational and integral in that letter.

57. The **degree** of a rational, integral equation in a given letter is the highest exponent of that letter in the equation.

In determining the degree of an equation according to this definition it is necessary that all indicated multiplications be performed as far as possible.

E.g. $(x-2)(x-3)=0$ is of the 2d degree in x, since it reduces to $x^2-5x+6=0$.

EQUIVALENT EQUATIONS

58. Two equations are said to be **equivalent** if every root of either is also a root of the other.

In the Elementary Course, § 36, we found that an equation may be changed into an equivalent equation by certain operations, which are now further considered in principles **1**, **2**, and **3** below:

59. **Principle 1.** *If one rational, integral equation is derived from another by performing the indicated operations, then the two equations are equivalent.*

This is evident, since in performing the indicated operations each expression is replaced by another identically equal to it. Hence any expression which satisfies the given equation must satisfy the other, and conversely.

E.g. $10x=50$ is equivalent to $3x+7x=50$, since $3x+7x\equiv 10x$; and $8(2x-3y)=2y-1$ is equivalent to $16x-24y=2y-1$, since $8(2x-3y)\equiv 16x-24y$.

60. **Principle 2.** *If any equation is derived from another by adding the same expression to each member, or by subtracting the same expression from each member, then the equations are equivalent.*

For simplicity consider an equation,

$$M=N, \qquad (1)$$

containing only one unknown, x. Add to each member an expression A, which may or may not contain x.

Then $$M + A = N + A \tag{2}$$

is easily seen to have the same roots as (1). For if a certain value of x makes M equal N, this will also make $M + A$ equal $N + A$, since any value whatever of x makes A equal A. Hence any number which is a root of (1) is also a root of (2).

Again, any value of x which makes $M + A = N + A$ will also make M equal N, since every value of x makes A equal A. Hence any number which is a root of (2) is also a root of (1). We have therefore shown that (1) and (2) are equivalent according to the definition, § 58.

This argument is based on axioms I and IV. By use of the same axioms we may show that $M - A = N - A$ and $M = N$ are equivalent equations.

61. It follows that *any equation can be reduced to an equivalent equation of the form* $\boldsymbol{R = 0}$.

For if an equation is in the form $M = N$, then by principle 2 it is equivalent to $M - N = N - N = 0$, which is in the form $R = 0$.

62. Principle 3. *If one equation is derived from another by multiplying or dividing each member by the same expression, then the equations are equivalent, provided the original equation is not multiplied or divided by zero or by an expression containing any of the unknowns of the equation.*

This principle follows from axioms V and IX by argument similar to that used in § 60. In this case, however, the expression A must not contain x and must not be zero, as was possible in principle **2**.

E.g. $x + 1 = 5$ and $(x - 1)(x + 1) = 5(x - 1)$ are not equivalent equations, since the first has only the root $x = 4$, while the second has in addition the root $x = 1$, as may be easily verified.

Similarly, $x - 5 = 7$ and $0 \cdot (x - 5) = 0 \cdot 7$ are not equivalent, since the first has only the root $x = 12$, while the second is satisfied by any number whatever.

63. The ordinary processes of solving equations depend upon principles **1**, **2**, and **3**, as is illustrated by the following examples:

Ex. 1. $$(x+4)(x+5)=(x+2)(x+6). \quad (1)$$

$$x^2+9x+20=x^2+8x+12. \quad (2)$$

$$x=-8. \quad (3)$$

By principle **1**, (1) and (2) are equivalent, and by principle **2**, (2) and (3) are equivalent. Hence (1) and (3) are equivalent. That is, -8 is the solution of (1).

Ex. 2. $$\tfrac{2}{3}x+\tfrac{4}{3}=4. \quad (1)$$

$$2x+4=12. \quad (2)$$

$$2x=8. \quad (3)$$

$$x=4. \quad (4)$$

By principle **3**, (1) and (2) are equivalent. By principle **2**, (2) and (3) are equivalent. By **3**, (3) and (4) are equivalent. Hence (1) and (4) are equivalent and 4 is the solution of (1).

These principles are stated for equations, but they apply equally well to identities, inasmuch as the identities are changed into other identities by these operations.

64. If an *identity* is reduced to the form $R=0$, § 61, and all the indicated operations are performed, then it becomes $0=0$. See § 53. Conversely, if an equality may be reduced to the form $0=0$, it is an identity. This, therefore, is a *test* as to whether an equality is an identity.

E.g. $(x+4)^2=x^2+8x+16$ is an identity, since in $x^2+8x+16-x^2-8x-16=0$ all terms cancel, leaving $0=0$.

EXERCISES

In the following, determine which numbers or sets of numbers, if any, of those written to the right, satisfy the corresponding equation.

Remember that no substitution is legitimate which reduces any denominator to zero.

1. $4(x-1)(x-2)(x-3)=3(x-2)(x-3)$. 1, 2, 3, 4.

2. $\dfrac{x^2-16}{x+5}=(x-4)(x+6)$. 2, 4, 6.

3. $\dfrac{x+3}{\sqrt{x^2+7}}=x-\dfrac{3}{2}$. 2, 3, $\frac{1}{2}$.

4. $\dfrac{(x-3)(x-2)}{x^2-7x+10}=x^2-5x+6$. 2, 3, 0, -2.

5. $\dfrac{a^2+9a+20}{a^2+8a+16}=(a+4)(a-4)(a+5)$. 4, -4, 5, -5.

6. $3a+4b=12$. $\begin{cases}a=0,\\b=3.\end{cases}$ $\begin{cases}a=4,\\b=0.\end{cases}$ $\begin{cases}a=2,\\b=2.\end{cases}$

7. $\dfrac{369(a-b)}{a^2+b^2}=a+b$. $\begin{cases}a=0,\\b=0.\end{cases}$ $\begin{cases}a=1,\\b=1.\end{cases}$ $\begin{cases}a=5,\\b=4.\end{cases}$

8. $\dfrac{x^2-y^2}{x-y}=(x-2)(y-1)$. $\begin{cases}x=1,\\y=0.\end{cases}$ $\begin{cases}x=1,\\y=1.\end{cases}$ $\begin{cases}x=2,\\y=2.\end{cases}$

9. $\dfrac{u^3-v^3}{u^2-v^2}=(u^2+uv+v^2)(u-v)$. $\begin{cases}u=1,\\v-1.\end{cases}$ $\begin{cases}u=1,\\v=0.\end{cases}$ $\begin{cases}u=-1,\\v=0.\end{cases}$

10. $\dfrac{(r-s)(r+s)(r^2+s^2)}{r^3+rs^2-r^2s-s^3}=(r^2-s^2)(2r-3s)$. $r=1,\ s=1$;
$r=1,\ s=-1$; $r=2,\ s=-2$; $r=a,\ s=-a$.

11. $a+b+c=6$. $a=1,\ b=2,\ c=3$;
$a=3,\ b=3,\ c=0$; $a=10,\ b=0,\ c=-4$.

12. $\dfrac{a-b+c}{\sqrt{a^2+b^2+c^2}}=\dfrac{3a-2c+5b+2}{10}$. $a=8,\ b=0,\ c=6$;
$a=1,\ b=4,\ c=2$; $a=0,\ b=0,\ c=-4$.

13. $\dfrac{(a-b)(b-c)(c-a)}{ac-bc-a^2+ba}=(b-b)(c-a)$. $a=2,\ b=1,\ c=1$;
$a=3,\ b=2,\ c=3$; $a=6,\ b=6,\ c=0$.

14. $(x-z)(x-y)(y-z)=8\,xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)$.
$x=1,\ y=1,\ z=1;\quad x=1,\ y=0,\ z=1;\quad x=1,\ y=2,\ z=3.$

15. $x^3+3\,x^2y+3\,xy^2+y^3=(x+y)^3.\ \begin{cases}x=1,\\ y=1.\end{cases}\ \begin{cases}x=1,\\ y=2.\end{cases}\ \begin{cases}x=3,\\ y=4.\end{cases}$

16. Show by reducing the equality in Ex. 15 to the form $R=O$ that it is satisfied by any pair of values whatsoever for x and y, *e.g.*, for $x=348764$, $y=594021$. What kind of an equality is this?

Which of the following four equalities are identities?

17. $12(x+y)^2+17(x+y)-7=(3\,x+3\,y-1)(4\,x+4\,y+7)$.

18. $\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4$.

19. $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$.

20. $2(a-b)^2+5(a+b)+8\,ab=(2\,a+2\,b+1)(a+b+1)$.

Solve the following equations, and verify results in 21–25:

21. $(2\,a+3)(3\,a-2)=a^2+a(5\,a+3)$.

22. $6(b-4)^2=-5-(3-2\,b)^2-5(2+b)(7-2\,b)$.

23. $(y-3)^2+(y-4)^2-(y-2)^2-(y-3)^2=0$.

24. $(x-3)(3\,x+4)-(x-4)(x-2)=(2\,x+1)(x-6)$.

25. $2(3\,r-2)(4\,r+1)+(r-4)^3=(r+4)^3-2$.

26. $a^3-c+b^3c+abc=b$. (Solve for c.)

27. $(b-2)^2(b-y)-3\,by+(2\,b+1)(b-1)=3-2\,b$. (Find y.)

28. $2(12-x)+3(5\,x-4)+2(16-x)=12(3+x)$.

29. $(b-a)x-(a+b)x+4\,a^2=0$. (Find x.)

30. $(x-a)(b-c)+(b-a)(x-c)-(a-c)(x-b)=0$. (Find x.)

31. $r^3v+s^3v-3\,r-3\,s+3\,v(r^2s+rs^2)=0$. (Find v.)

32. $(x-3)(x-7)-(x-5)(x-2)+12=2(x-1)$.

33. $(a+b)^2+(x-b)(x-a)-(x+a)(x+b)=0$. (Find x.)

34. $ny(y+n)-(y+m)(y+n)(m+n)+my(y+m)=0$. (Find y.)

35. $(n+i)(j-i+k)-(n-i)(i-j+k)=0$. (Find n.)

36. $\frac{7}{12}(5x-1)+\frac{5}{16}(2-3x)+\frac{1}{8}(4+x)=\frac{3}{8}(1+2x)-\frac{9}{16}$.

37. $(l-m)(z-n)+2l(m+n)=(l+m)(z+n)$. (Find z.)

38. $a(x-b)-(a+b)(x+b-a)=b(x-a)+a^2-b^2$. (Find x.)

39. $(m+n)(n+b-y)+(n-m)(b-y)=n(m+b)$. (Find y.)

40. $\frac{3(2a-3b)}{8}-\frac{2(3a-5b)}{3}+\frac{5(a-b)}{6}=\frac{b}{8}$. (Find a.)

Solve each of the following equations for each letter in terms of the others.

41. $l(W+w')=l'W'$.

42. $(v-n)d=(v-n_1)d_1$.

43. $m_2s_2(t_2-t)=(m+m_1)(t-t_1)$.

44. $(m+m_1)(t_1-t)=lm_2+m_2t$.

PROBLEMS

1. What number must be added to each of the numbers 2, 26, 10 in order that the product of the first two sums may equal the square of the last sum?

2. What number must be subtracted from each of the numbers 9, 12, 18 in order that the product of the first two remainders may equal the square of the last remainder?

3. What number must be added to each of the numbers a, b, c in order that the product of the first two sums may equal the square of the last?

Note that problem 1 is a special case of 3. Explain how 2 may also be made a special case of 3.

4. What number must be added to each of the numbers a, b, c, d in order that the product of the first two sums may equal the product of the last two?

5. State and solve a problem which is a special case of problem 4.

6. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

7. State and solve a problem which is a special case of problem 6.

8. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may be k more than twice the product of the last two?

9. State and solve a problem which is a special case of problem 8.

10. The radius of a circle is increased by 3 feet, thereby increasing the area of the circle by 50 square feet. Find the radius of the original circle.

The area of a circle is πr^2. Use $3\frac{1}{7}$ for π.

11. The radius of a circle is decreased by 2 feet, thereby decreasing the area by 36 square feet. Find the radius of the original circle.

12. State and solve a general problem of which 10 is a special case.

13. State and solve a general problem of which 11 is a special case.

How may the problem stated under 12 be interpreted so as to include the one given under 13?

14. Each side of a square is increased by a feet, thereby increasing its area by b square feet. Find the side of the original square.

Interpret this problem if a and b are both negative numbers.

15. State and solve a problem which is a special case of 14, (1) when a and b are both positive, (2) when a and b are both negative.

16. Two opposite sides of a square are each increased by a feet and the other two by b feet, thereby producing a rectangle whose area is c square feet greater than that of the square. Find the side of the square.

Interpret this problem when a, b, and c are all negative numbers.

17. State and solve a problem which is a special case of 16, (1) when a, b, and c are all positive, (2) when a, b, and c are all negative.

18. A messenger starts for a distant point at 4 A.M., going 5 miles per hour. Four hours later another starts from the same place, going in the same direction at the rate of 9 miles per hour. When will they be together? When will they be 8 miles apart? How far apart will they be at 2 P.M.?

For a general explanation of problems on motion, see p. 115, E. C.

19. One object moves with a velocity of v_1 feet per second and another along the same path in the same direction with a velocity of v_2 feet. If they start together, how long will it require the latter to gain n feet on the former?

From formula (2), p. 117, E. C., we have $t = \dfrac{n}{v_2 - v_1}$.

Discussion. If $v_2 > v_1$ and $n > 0$, the value of t is positive, *i.e.* the objects will be in the required position some time *after* the time of starting.

If $v_2 < v_1$ and $n > 0$, the value of t is negative, which may be taken to mean that if the objects had been moving before the instant taken in the problem as the time of starting, then they would have been in the required position some time *earlier*.

If $v_2 = v_1$ and $n \neq 0$, the solution is impossible. See § 25. This means that the objects will never be in the required position. If $v_1 = v_2$ and $n = 0$, the solution is indeterminate. See § 24. This may be interpreted to mean that the objects are always in the required position.

20. State and solve a problem which is a special case of 19 under each of the conditions mentioned in the discussion.

21. At what time after 5 o'clock are the hands of a clock first in a straight line?

22. Saturn completes its journey about the sun in 29 years and Uranus in 84 years. How many years elapse from conjunction to conjunction? See figure, p. 119, E. C.

23. An object moves in a fixed path at the rate of v_1 feet per second, and another which starts a seconds later moves in the same path at the rate of v_2 feet per second. In how many seconds will the latter overtake the former?

24. In problem 23 how long before they will be d feet apart?

If in problem 24 d is zero, this problem is the same as 23. If d is not zero and a is zero, it is the same as problem 19.

25. A beam carries 3 weights, one at each end weighing 100 and 120 pounds respectively, and the third weighing 150 pounds 2 feet from its center, where the fulcrum is. What is the length of the beam if this arrangement makes it balance?

For a general explanation of problems involving the lever, see pp. 120–122, E. C.

26. A beam whose fulcrum is at its center is made to balance when weights of 60 and 80 pounds are placed at one end and 2 feet from that end respectively, and weights of 50 and 100 pounds are placed at the other end and 3 feet from it respectively. Find the length of the beam.

27. How many cubic centimeters of matter, density 4.20, must be added to 150 ccm. of density 8.10 so that the density of the compound shall be 5.4? See § 99, E. C.

28. How many cubic centimeters of nitrogen, density 0.001255, must be mixed with 210 ccm. of oxygen, density 0.00143, to form air whose density is 0.001292?

29. A man can do a piece of work in 16 days, another in 18 days, and a third in 15 days. How many days will it require all to do it when working together?

30. A can do a piece of work in a days, B can do it in b days, C in c days, and D in d days. How long will it require all to do it when working together?

CHAPTER IV

INTEGRAL LINEAR EQUATIONS IN TWO OR MORE UNKNOWNS

INDETERMINATE EQUATIONS

65. If a single equation contains two unknowns, an **unlimited number** of pairs of numbers may be found which satisfy the equation.

E.g. In the equation, $y = 2x + 1$, by assigning any value to x, a corresponding value of y may be found such that the two together satisfy the equation.

Thus, $x = -3$, $y = -5$; $x = 0$, $y = 1$; $x = 2$, $y = 5$, are pairs of numbers which satisfy this equation.

For this reason a single equation in two unknowns is called an **indeterminate** equation, and the unknowns are called **variables**. A **solution** of such an equation is any pair of numbers which satisfy it.

A picture or map of the real (see §§ 135, 195) solutions of an indeterminate equation in two variables may be made by means of the **graph** as explained in §§ 107, 108, E. C.

66. The **degree** of an equation in two or more letters is the sum of the exponents of those letters in that one of its terms in which this sum is greatest. See § 110, E. C.

E.g. $y = 2x + 1$ is of the *first degree* in x and y. $y^2 = 2x + y$ and $y = 2xy + 3$ are each of the *second degree* in x and y.

An equation of the first degree in two variables is called a **linear** equation, since it can be shown that the graph of every such equation is a straight line.

67. It is often important to determine those solutions of an indeterminate equation which are **positive integers**, and for this purpose the graph is especially useful.

Ex. 1. Find the positive integral solutions of the equation

$$3x + 7y = 42.$$

Solution. Graph the equation carefully on cross-ruled paper, finding it to cut the x-axis at $x = 14$ and the y-axis at $y = 6$.

Look now for the *corner points* of the unit squares through which this straight line passes. The coördinates of these points, if there are such points, are the solutions required. In this case the line passes through only one such point, namely the point (7, 3). Hence the solution sought is $x = 7$, $y = 3$.

Ex. 2. Find the *least* positive integers which satisfy

$$7x - 3y = 17.$$

Solution. This line cuts the x-axis at $x = 2\frac{3}{7}$ and the y-axis at $y = -5\frac{2}{3}$. On locating these points as accurately as possible, the line through them *seems* to cut the corner points (5, 6) and (8, 13). The coördinates of both these points satisfy the equation. Hence the solution sought is $x = 5$, $y = 6$.

EXERCISES

Solve in positive integers by means of graphs, and check:

1. $x + y = 7.$
2. $x + y = 3.$
3. $x - 27 = -9y.$
4. $7y - 112 = -4x.$
5. $90 - 5x = 9y.$
6. $5x = 29 - 3y.$
7. $140 - 7x - 10y = 0.$
8. $8 - 2x - y = 0.$

Solve in least positive integers, and check:

9. $7x = 3y + 21.$
10. $5x - 4y = 20.$
11. $4x = 9y - 36.$
12. $5x - 2y + 10 = 0.$

68. In the case of two indeterminate equations, each of the first degree in two variables, the coördinates of the point of intersection of their graphs form a solution of *both equations.*

Since these graphs are straight lines, they have *only one point* in common, and hence there is only *one solution* of the given pair of equations.

E.g. On graphing $x + y = 4$ and $y - x = 2$, the lines are found to intersect in the point (1, 3). Hence the solution of this pair of equations is

$$x = 1, y = 3.$$

EXERCISES

Graph the following and thus find the solution of each pair of equations. Check by substituting in the equations.

1. $\begin{cases} 3x - 2y = -2, \\ x + 7y = 30. \end{cases}$

2. $\begin{cases} x + y = 2, \\ 3x + 2y = 3. \end{cases}$

3. $\begin{cases} x - 4y = 1, \\ 2x - y = -5. \end{cases}$

4. $\begin{cases} x = -1, \\ 2x - 3y = 1. \end{cases}$

5. $\begin{cases} 4x = 2y + 6, \\ x - 5 = y - 1. \end{cases}$

6. $\begin{cases} x = y - 5, \\ 5y = x + 9. \end{cases}$

7. $\begin{cases} 8x = 7y, \\ x + 3 = 5y + 3. \end{cases}$

8. $\begin{cases} y = 1, \\ 3y + 4x = y. \end{cases}$

9. $\begin{cases} 2x - 4y = 4, \\ x - y = 6y - 3. \end{cases}$

10. $\begin{cases} x = 4, \\ y + x = 8. \end{cases}$

11. $\begin{cases} y = -3, \\ 3x + 2y = 3. \end{cases}$

12. $\begin{cases} x = -2, \\ y = 5. \end{cases}$

SOLUTION BY ELIMINATION

69. The solution of a pair of equations such as the foregoing may be obtained without the use of the graph by the process called **elimination**. See pages 154–159, E. C.

70. Elimination by **substitution** consists in expressing one variable in terms of the other in one equation and substituting this result in the other equation, thus obtaining an equation in which only one variable appears. See § 116, E. C.

71. Elimination by **addition or subtraction** consists in making the coefficients of one variable the same in the two equations (§ 62), so that when the members are added or subtracted this variable will not appear in the resulting equation. See § 117, E. C.

72. Elimination by **comparison** is a third method, which consists in expressing the same variable in terms of the other in each equation and equating these two expressions to each other.

As an example of elimination by comparison, solve

$$\begin{cases} 3y + x = 14, & (1) \\ 2y - 5x = -19. & (2) \end{cases}$$

From (1), $$x = 14 - 3y. \qquad (3)$$

From (2), $$x = \frac{19 + 2y}{5}. \qquad (4)$$

From (3) and (4), $$14 - 3y = \frac{19 + 2y}{5}. \qquad (5)$$

Solving (5), $$y = 3.$$

Substituting in (1), $$x = 5.$$

Check by substituting $x = 5$, $y = 3$ in both (1) and (2).

In applying any method of elimination it is desirable first to reduce each equation to the standard form: $ax + by = c$. See § 119, E. C.

EXERCISES

Solve the following pairs of equations by one of the processes of elimination.

1. $\begin{cases} 3x + 2y = 118, \\ x + 5y = 191. \end{cases}$

2. $\begin{cases} 5x - 8\frac{1}{2} = 7y - 44, \\ 2x = y + \frac{5}{7}. \end{cases}$

3. $\begin{cases} 6x - 3y = 7, \\ 2x - 2y = 3. \end{cases}$

4. $\begin{cases} 3x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x, \\ 2\frac{1}{2}x + \frac{1}{2}y = 1. \end{cases}$

5. $\begin{cases} 5x - 11y - 2 = 4x, \\ 5x - 2y = 63. \end{cases}$

6. $\begin{cases} 3y + 40 = 2x + 14, \\ 9y - 347 = 5x - 420. \end{cases}$

7. $\begin{cases} 5y - 3x + 8 = 4y + 2x + 7, \\ 4x - 2y = 3y + 2. \end{cases}$

8. $\begin{cases} 6y - 5x = 5x + 14, \\ 3y - 2x - 6 = 5 + x. \end{cases}$

9. $\begin{cases}(x+5)(y+7)=(x+1)(y-9)+112,\\ 2x+10=3y+1.\end{cases}$

10. $\begin{cases}73-7y=5x,\\ 2y-3x=12.\end{cases}$

11. $\begin{cases}ax=by,\\ x+y=c.\end{cases}$

12. $\begin{cases}\dfrac{x}{a}=\dfrac{y}{b},\\ x+y=s.\end{cases}$

13. $\begin{cases}x+y=a,\\ x-y=b.\end{cases}$

14. $\begin{cases}ax+by=c,\\ fx+gy=h.\end{cases}$

15.* $\begin{cases}\dfrac{3}{x}-\dfrac{5}{y}=6,\\ \dfrac{2}{x}+\dfrac{3}{y}=2.\end{cases}$

16.* $\begin{cases}\dfrac{a}{x}+\dfrac{b}{y}=c,\\ \dfrac{f}{x}+\dfrac{g}{y}=h.\end{cases}$

SOLUTION BY FORMULA

73. We now proceed to a more general study of a pair of linear equations in two variables.

Ex. 1. Solve
$$\begin{cases}2x+3y=4, & (1)\\ 5x+6y=7. & (2)\end{cases}$$

Multiplying (1) by 5 and (2) by 2,
$$5\cdot 2x+5\cdot 3y=5\cdot 4, \qquad (3)$$
$$2\cdot 5x+2\cdot 6y=2\cdot 7. \qquad (4)$$

Subtracting (3) from (4), $(2\cdot 6-5\cdot 3)y=2\cdot 7-5\cdot 4.$ (5)

Solving for y,
$$y=\frac{2\cdot 7-5\cdot 4}{2\cdot 6-5\cdot 3}=\frac{-6}{-3}=2. \qquad (6)$$

In like manner, solving for x by eliminating y,

we have
$$x=\frac{4\cdot 6-7\cdot 3}{2\cdot 6-5\cdot 3}=\frac{3}{-3}=-1. \qquad (7)$$

Ex. 2. In this manner, solving,
$$\begin{cases}7x+9y=71,\\ 2x+3y=48,\end{cases}$$

we find
$$x=\frac{71\cdot 3-48\cdot 9}{7\cdot 3-2\cdot 9} \text{ and } y=\frac{7\cdot 48-2\cdot 71}{7\cdot 3-2\cdot 9}.$$

* Let $\dfrac{1}{x}$ and $\dfrac{1}{y}$ be the unknowns.

In Ex. 2, the various coefficients are found to occupy the *same relative positions* in the expressions for x and y as the corresponding coefficients do in Ex. 1.

Show that this is also true in the following:

Ex. 3. $\begin{cases} 3x + 7y = 10, \\ 2x - 5y = 7. \end{cases}$ Ex. 4. $\begin{cases} 5x - 3y = 8, \\ 2x + 7y = 19. \end{cases}$

A convenient rule for reading directly the values of the unknowns in such a pair of equations may be made from the solution of the following:

Ex. 5. Solve $\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$

Eliminating first y and then x as in Ex. 1, we find:

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

To remember these results, notice that the coefficients of x and y in the given equations stand in the form of a square, thus $\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}$, and that the denominator in the expressions for both x and y is the *cross product* a_1b_2 minus the *cross product* a_2b_1. The numerator in the expression for x is read by replacing the a's in this square by the c's, *i.e.*, $\begin{matrix} c_1 & b_1 \\ c_2 & b_2 \end{matrix}$, and then reading the cross products as before. The numerator for y is read by replacing the b's by the c's, *i.e.*, $\begin{matrix} a_1 & c_1 \\ a_2 & c_2 \end{matrix}$, and then reading the cross products.

74. To indicate that the coefficients in a pair of equations are to be treated as just described, we write $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$ and call $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ a **determinant.** These are much used in higher algebra.

Since any pair of linear equations in two unknowns may be reduced to the standard form as given in Ex. 5, it follows that the values of x and y there obtained constitute a *formula for the solution of any pair of such equations.*

EXERCISES

Reduce the following pairs of equations to the standard form and write out the solutions by the formula:

1. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

2. $\begin{cases} 4x - 5y = -26, \\ 2x - 3y = -14. \end{cases}$

3. $\begin{cases} 6y - 17 = -5x, \\ 6x - 16 = -5y. \end{cases}$

4. $\begin{cases} \frac{1}{4}(x-3) = -\frac{1}{3}(y-2) + \frac{1}{2}x, \\ \frac{1}{2}(y-1) = x - \frac{1}{8}(x-2). \end{cases}$

5. $\begin{cases} 2x - y = 53, \\ 19x + 17y = 0. \end{cases}$

6. $\begin{cases} ax - by = 0, \\ x - y = c. \end{cases}$

7. $\begin{cases} mx + ny = p, \\ rx + sy = t. \end{cases}$

8. $\begin{cases} a(x+y) - b(x-y) = 2a, \\ a(x-y) - b(x+y) = 2b. \end{cases}$

9. $\begin{cases} (k+1)x + (k-2)y = 3a, \\ (k+3)x + (k-4)y = 7a. \end{cases}$

10. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

11. $\begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2a^2 - 2b^2. \end{cases}$

12. $\begin{cases} \frac{1}{2}(a-b) - \frac{1}{5}(a-3b) = b - 3, \\ \frac{3}{4}(a-b) + \frac{5}{6}(a+b) = 18. \end{cases}$

13. $\begin{cases} a(x+y) + b(x-y) = 2, \\ a^2(x+y) - b^2(x-y) = a - b. \end{cases}$

14. $\begin{cases} 7(x-5) = 3 - \dfrac{y}{2} - x, \\ \frac{1}{4}(x-y) + \frac{1}{2}y - \frac{5}{8}(x-1) = -1. \end{cases}$

15. $\begin{cases} mx + ny = m^3 + 2m^2n + n^3, \\ nx + my = m^3 + 2mn^2 + n^3. \end{cases}$

16. $\begin{cases} (m+n)x - (m-n)y = 2lm, \\ (m+l)x - (m-l)y = 2mn. \end{cases}$

17. $\begin{cases} (a-b)x + (a+b)y = 2a, \\ (a-b)x - (a+b)y = 2b. \end{cases}$

18. $\begin{cases} (h+k)x + (h-k)y = 2(h^2+k^2), \\ (h-k)x + (h+k)y = 2(h^2-k^2). \end{cases}$

19. $\begin{cases} (a-b)x + y(a^2+b^2) = (a+b)^2 + a + b - 2ab, \\ (b-a)x + y(a^2+b^2) = a + b - a^2 - b^2. \end{cases}$

INCONSISTENT AND DEPENDENT EQUATIONS

75. A pair of linear equations in two variables may be such that they either have no solution or have an unlimited number of solutions.

Ex. 1. Solve
$$\begin{cases} x - 2y = -2, & (1) \\ 3x - 6y = -12, & (2) \end{cases}$$

On graphing these equations they are found to represent two parallel lines. Since the lines have no point in common, it follows that the equations have no solution. See Fig. 1.

Attempting to solve them by means of the formula, § 73, we find:

$$x = \frac{(-2)(-6)-(-12)(-2)}{1(-6)-3(-2)} = \frac{-12}{0},$$

and
$$y = \frac{1(-12)-3(-2)}{1(-6)-3(-2)} = \frac{-6}{0}.$$

But by § 25, $\frac{-12}{0}$ and $\frac{-6}{0}$ *are not numbers*. Hence, from this it follows that the given equations have no solution. In this case no solution is **possible**, and the equations are said to be **contradictory**.

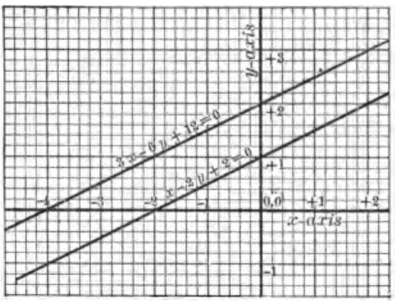


FIG. 1.

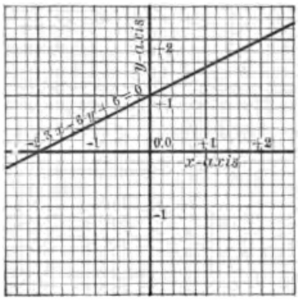


FIG. 2.

Ex. 2. Solve
$$\begin{cases} 3x - 6y = -6, & (1) \\ x - 2y = -2, & (2) \end{cases}$$

On graphing these equations, they are found to represent the *same line.* Hence every pair of numbers satisfying one equation must satisfy the other also. See Fig. 2.

Solving these equations by the formula, we find:

$$x=\frac{(-6)(-2)-(-2)(-6)}{3(-2)-1(-6)}=\frac{0}{0} \text{ and } y=\frac{3(-2)-1(-6)}{3(-2)-1(-6)}=\frac{0}{0}.$$

But by § 24, $\frac{0}{0}$ may represent *any number whatever*. Hence we may select for one of the unknowns any value we please and find from (1) or (2) a corresponding value for the other, but we may not select arbitrary values for *both* x and y.

In this case the solution is **indeterminate** and the equations are **dependent**; that is, one may be derived from the other.

Thus, (2) is derived from (1) by dividing both members by 3.

76. Two linear equations in two variables which have one and only one solution are called **independent and consistent.**

The cases in which such pairs of equations are *dependent* or *contradictory* are those in which the denominators of the expressions for x and y become *zero*. Hence, in order that such a pair of equations may have a *unique* solution, the denominator $a_1b_2-a_2b_1$ of the formula, § 73, *must not reduce to zero*. This may be used as a test to determine whether a given pair of equations is independent and consistent.

EXERCISES

In the following, show both by the formula and by the graph which pairs of equations are independent and consistent, which dependent, and which contradictory.

1. $\begin{cases} 5x-3y=5, \\ 5x-3y=9. \end{cases}$

2. $\begin{cases} x-7+5y=y-x-2, \\ 5x+3y-4=3x-y+3. \end{cases}$

3. $\begin{cases} 7x-3y-4=2x-2, \\ x+y-3=2x-7. \end{cases}$

4. $\begin{cases} x-3y=6, \\ 5x-15y=18. \end{cases}$

5. $\begin{cases} 3y-4x-1=2x-5y+8, \\ 2y-5x+8=3x+y. \end{cases}$

6. $\begin{cases} 3x-6y+5=2x-5y+7, \\ 5x+3y-1=3x+5y+3. \end{cases}$

7. $\begin{cases} 2y+7x=2+6x, \\ 4x-3y=4+3x-5y. \end{cases}$

8. $\begin{cases} 5x-3=7y+8, \\ 2x+7=4y-9. \end{cases}$

9. $\begin{cases} 5x+2y=6+3x+5y, \\ 3x+y=18-3x+10y. \end{cases}$

10. $\begin{cases} 3x+4y=7+5y, \\ x-y=6-2x. \end{cases}$

SYSTEMS OF EQUATIONS IN MORE THAN TWO UNKNOWNS

77. If a single linear equation in three or more variables is given, there is no limit to the number of sets of values which satisfy it.

E.g. $3x + 2y + 4z = 24$ is satisfied by $x = 1$, $y = 3$, $z = 3\frac{3}{4}$; $x = 2$, $y = 2$, $z = 3\frac{1}{2}$; $x = 0$, $y = 0$, $z = 6$; etc.

If two linear equations in three or more variables are given, they have in general an unlimited number of solutions.

E.g. $3x + 2y + 4z = 24$ and $x + y + z = 6$ are both satisfied by $x = 2$, $y = -1$, $z = 5$; $x = 3$, $y = -1\frac{1}{2}$, $z = 4\frac{1}{2}$; etc.

But if a system of linear equations contains *as many equations as variables*, it has in general one and only one set of values which satisfy all the equations.

E.g. The system $\begin{cases} x + y + z = 6, \\ 3x - y + 2z = 7, \\ 2x + 3y - z = 5, \end{cases}$

is satisfied by $x = 1$, $y = 2$, $z = 3$, and by no other set of values.

It may happen, however, as in the case of two variables, that such a system is not *independent* and *consistent*.

Such cases frequently occur in higher work, and a general rule is there found for determining the nature of a system of linear equations *without solving them*; namely, by means of *determinants* (§ 73). In this book the only test used is the result of the solution itself as explained in the next paragraph.

78. An independent and consistent system of linear equations in three variables may be solved as follows:

From two of the equations, say the 1st and 2d, eliminate one of the variables, obtaining *one* equation in the remaining two variables.

From the 1st and 3d equations eliminate the same variable, obtaining a *second* equation in the remaining two variables.

Solve as usual the two equations thus found. Substitute the values of these two variables in one of the given equations, and thus find the value of the third variable.

The process of elimination by addition or subtraction is usually most convenient. See § 120, E. C.

EXERCISES

Solve each of the following systems and check by substituting in each equation:

1. $\begin{cases} 2x + 5y - 7z = 9, \\ 5x - y + 3z = 16, \\ 7x + 6y + z = 34. \end{cases}$

2. $\begin{cases} a + b + c = 9, \\ 8a + 4b + 2c = 36, \\ 27a + 9b + 3c = 93. \end{cases}$

3. $\begin{cases} 18l - 7m - 5n = 161, \\ 4\frac{2}{5}m - \frac{2}{3}l + n = 18, \\ 3\frac{1}{2}n + 2m + \frac{3}{4}l = 33. \end{cases}$

4. $\begin{cases} \dfrac{a}{3} + \dfrac{b}{6} + \dfrac{c}{9} = -2, \\ \dfrac{a}{6} + \dfrac{b}{9} + \dfrac{c}{12} = -4, \\ \dfrac{a}{9} + \dfrac{b}{12} + \dfrac{c}{15} = -4. \end{cases}$

5. $\begin{cases} 8z - 3y + x = -2, \\ 3x - 5y - 6z = -46, \\ y + 5x - 4z = -18. \end{cases}$

6.* $\begin{cases} \dfrac{3}{a} = \dfrac{2}{b}, \\ \dfrac{2}{a} + \dfrac{5}{b} - \dfrac{4}{c} = 17, \\ \dfrac{7}{a} - \dfrac{3}{b} + \dfrac{6}{c} = 8. \end{cases}$

7. $\begin{cases} x + 2y - 3z = -3, \\ 2x - 3y + z = 8, \\ 5x - 4y - 7z = -5. \end{cases}$

8. $\begin{cases} 2x + 3y - 7z = 19, \\ 5x + 8y + 11z = 24, \\ 7x + 11y + 4z = 43. \end{cases}$

Show that this system is not independent.

9. $\begin{cases} x + y = 16, \\ z + x = 22, \\ y + z = 28. \end{cases}$

10. $\begin{cases} x + 2y = 26, \\ 3x + 4z = 56, \\ 5y + 6z = 65. \end{cases}$

11. $\begin{cases} a + b + c = 5, \\ 3a - 5b + 7c = 79, \\ 9a - 11b = 91. \end{cases}$

12. $\begin{cases} l + m + n = 29\frac{1}{4}, \\ l + m - n = 18\frac{1}{4}, \\ l - m + n = 13\frac{3}{4}. \end{cases}$

* Use $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ as the unknowns.

13. $\begin{cases} l+m+n=a, \\ l+m-n=b, \\ l-m+n=c. \end{cases}$

14. $\begin{cases} ax+by=p, \\ cy+dz=q, \\ ex+fz=r. \end{cases}$

15. $\begin{cases} \dfrac{1}{a}+\dfrac{1}{b}=4, \\ \dfrac{1}{a}+\dfrac{1}{c}=3, \\ \dfrac{1}{b}+\dfrac{1}{c}=2. \end{cases}$

16. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x}+\dfrac{1}{z}=b, \\ \dfrac{1}{y}+\dfrac{1}{z}=c. \end{cases}$

17. $\begin{cases} u+2v+3x+4y=30, \\ 2u+3v+4x+5y=40, \\ 3u+4v+5x+6y=50, \\ 4u+5v+6x+7y=60. \end{cases}$

18. Make a rule for solving a system of four or more linear equations in as many variables as equations.

PROBLEMS INVOLVING TWO OR MORE UNKNOWNS

1. A man invests a certain amount of money at 4% interest and another amount at 5%, thereby obtaining an annual income of $3100. If the first amount had been invested at 5% and the second at 4%, the income would have been $3200. Find each investment.

2. The relation between the readings of the Centigrade and the Fahrenheit thermometers is given by the equation $F = 32 + \frac{9}{5}C$. The Fahrenheit reading at the melting temperature of osmium is 2432 degrees higher than the Centigrade. Find the melting temperature in each scale.

In the Réaumur thermometer the freezing and boiling points are marked 0° and 80° respectively. Hence if C is the Centigrade reading and R the Réaumur reading, then $R = \frac{4}{5}C$. See § 101, E. C.

3. What is the temperature Fahrenheit (*a*) if the Fahrenheit reading equals $\frac{1}{2}$ of the sum of the other two, (*b*) if the Centigrade reading equals $\frac{1}{2}$ of the Fahrenheit minus the Réaumur, (*c*) if the Réaumur is equal to the sum of the Fahrenheit and Centigrade?

4. Going with a current a steamer makes 19 miles per hour, while going against a current $\frac{3}{4}$ as strong the boat makes 5 miles per hour. Find the speed of each current and the boat.

5. There is a number consisting of 3 digits whose sum is 11. If the digits are written in reverse order, the resulting number is 594 less than the original number. Three times the tens' digit is one more than the sum of the hundreds' and the units' digit.

6. A certain kind of wine contains 20 % alcohol and another kind contains 28 %. How many gallons of each must be used to form 50 gallons of a mixture containing 21.6 % alcohol?

7. The area of a certain trapezoid of altitude 8 is 68. If 4 is added to the lower base and the upper base is doubled, the area is 108. Find both bases.

A trapezoid is a four-sided figure whose upper base, b_1, and lower base, b_2, are parallel, but the other two sides are not. If h is the perpendicular distance between the bases, then the area is $a = \frac{h}{2}(b_1 + b_2)$.

8. If on her second westward journey the *Lusitania* had made 1 knot more per hour, she would have crossed in 4 hours and 38 minutes less than she did. But if her speed had been 4 knots per hour less, she would have required 23 hours and 10 minutes longer. Find the time of her passage and her average speed if the length of her course was 2780 knots.

9. Aluminium bronze is an alloy of aluminium and copper. The densities of aluminium, copper, and aluminium bronze are 2.6, 8.9, and 7.69 respectively. How many ccm. of each metal are used in 100 ccm. of the alloy? See § 99, E. C.

10. Wood's metal, which is used in fire extinguishers on account of its low melting temperature, is an alloy of bismuth, lead, tin, and cadmium. In 120 pounds of Wood's metal, $\frac{1}{3}$ of the tin plus $\frac{1}{6}$ of the lead minus $\frac{1}{20}$ of the bismuth equals 7 pounds. If $\frac{1}{2}$ of the lead and $\frac{1}{5}$ of the tin be subtracted from the bismuth, the remainder is 42 pounds. Find the amount of each metal if 15 pounds of cadmium is used.

11. The upper base of a trapezoid is 6 and its area is 168. If $\frac{1}{3}$ the lower base is added to the upper, the area is 210. Find the altitude and the lower base.

12. A and B can do a piece of work in 18 days, B and C in 24 days, and C and A in 36 days. How long will it require each man, working alone, to do it, and how long will it require all working together?

13. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. How long will it require each to do it working alone?

14. A beam resting on a fulcrum balances when it carries weights of 100 and 130 pounds at its respective ends. The beam will also balance if it carries weights of 80 and 110 pounds respectively 2 feet from the ends. Find the distance from the fulcrum to the ends of the beam.

15. A beam carries three weights, A, B, and C. A balance is obtained when A is 12 feet from the fulcrum, B 8 feet from the fulcrum (on the same side as A), and C 20 feet from the fulcrum (on the side opposite A). It also balances when the distance of A is 8 feet, B 10 feet, and C 18 feet. Find the weights B and C if A is 50 lbs.

16. At 0° Centigrade sound travels 1115 feet per second with the wind on a certain day, and 1065 feet per second against the wind. Find the velocity of sound in calm weather, and the velocity of the wind on this occasion.

17. If the velocities of sound in air, brass, and iron at 0° Centigrade are x, y, z meters per second respectively, then $3x+2y-z=2505$, $5x-2y+z=151$, and $x+y+z=8777$. Find the velocity in each.

18. If x, y, z are the Centigrade readings at the temperatures which liquefy hydrogen, nitrogen, and oxygen respectively, then $3x-8y+2z=440$, $-8x+2y+4z=903$, and $x+4y-6z=60$. Find each temperature in both Centigrade and Fahrenheit readings.

19. Two trapezoids have a common lower base. Their altitudes are 8 and 10 respectively, and the sum of their areas is 148. If the upper base of the first trapezoid is multiplied by 2 and that of the second divided by 2, their combined area is 152; while if the upper base of the first is divided by 2 and that of the second multiplied by 2, the combined area is 176. Find the bases of each trapezoid.

20. If x, y, z are the Centigrade readings at the freezing temperatures of hydrogen, nitrogen, and oxygen respectively, then we have $x+y-3z=199$, $2x-5y+z=328$, and $-4x+2y+2z=156$. Find each temperature.

21. If x, y, z are respectively the melting point of carbon, the temperature of the hydrogen flame in air, and the temperature of this flame in pure oxygen, then $10x+2y+z=41{,}892$, $15x+y+2z=60{,}212$, and $7x+y+z=29{,}368$. Find each.

22. If a, b, c are the values in millions of the mineral products of the United States in 1880, 1900, and 1906 respectively, find each from the following relations:

$$5a-\frac{b}{8}-\frac{c}{10}=1572,\ \frac{a}{3}+\frac{b}{4}+\frac{c}{5}=669,\ a-\frac{b}{2}+\frac{c}{7}=37.$$

23. If x, y, z represent in thousands of tons the steel products of the United States in 1880, 1890, and 1905, find each from the following relations:

$$x+y+z=25{,}547,\ 3x+4y-z=826,\ x-3y+z=8439.$$

24. If the number of millions of tons of coal mined in the United States in 1890, 1900, and 1906 be represented by x, y, z respectively, find each from the following relations:

$$\frac{x}{2}+\frac{y}{30}+\frac{z}{25}=105,\ x-\frac{y}{9}+\frac{z}{17}=153,\ 3x+2y-2z=164.$$

25. If the values in millions of the farm products of the United States in 1870, 1900, and 1906 are represented by l, m, and n respectively, find each from the following relations:

$$2l+m-n=1633,\ 3l-2m+n=3440,\ l+m+n=13{,}675.$$

26. The sum of the areas of two trapezoids whose altitudes are 10 and 12 respectively is 284. If the upper base of the first is multiplied by 3 while its lower is decreased by 2, and the upper base of the second is divided by 2 while its lower base is increased by 3, the sum of the areas is 382; if the upper bases of both are doubled and the lower bases of both divided by 2, the sum of the areas is 322; and if the upper bases are divided by 2 while the first lower is doubled and the second trebled, the sum of the areas is 388. Find the bases of each trapezoid.

27. Two boys carry a 120-pound weight by means of a pole, at a certain point of which the weight is hung. One boy holds the pole 5 ft. from the weight and the other 3 ft. from it. What proportion of the weight does each boy lift?

Solution. Let x and y be the required amounts, then $5x$ is the leverage of the first boy and $3y$ that of the second, and these must be equal as in the case of the teeter, p. 122, E. C. Hence we have

$$5x = 3y, \text{ and } x + y = 120.$$

Solving, we find $x = 45$, $y = 75$.

28. If, in problem 27, the boys lift P_1 and P_2 pounds respectively at distances d_1 and d_2, and w is the weight lifted, then

$$P_1d_1 = P_2d_2. \tag{1}$$

$$P_1 + P_2 = w. \tag{2}$$

Solve (1) and (2), (*a*) when P_1 and P_2 are unknown, (*b*) when P_1 and w are unknown, (*c*) when P_1 and d_2 are unknown.

29. A weight of 540 pounds is carried on a pole by two men at distances of 4 and 5 feet respectively. How much does each lift?

30. A weight of 470 pounds is carried by two men, one at a distance of 3 feet and the other lifting 200 pounds. At what distance is the latter?

31. Two men are carrying a weight on a pole at distances of 4 and 6 feet respectively. The former lifts 240 pounds. How many pounds are they carrying?

CHAPTER V

FACTORING

79. A rational integral expression is said to be **completely factored** when it cannot be further resolved into factors which are rational and integral. Such factors are called **prime factors.**

The simpler forms of factoring are given in the following outline.

A **monomial factor** of any expression is evident at sight, and its removal should be the first step in every case.

E.g. $$4\,ax^2 + 2\,a^2x = 2\,ax(2\,x + a).$$

FACTORS OF BINOMIALS

80. *The difference of two squares.*

E.g. $$4\,x^2 - 9\,z^4 = (2\,x + 3\,z^2)(2\,x - 3\,z^2).$$

81. *The difference of two cubes.*

E.g. $$\begin{aligned} 8\,x^3 - 27\,y^3 &= (2\,x - 3\,y)[(2\,x)^2 + (2\,x)(3\,y) + (3\,y)^2] \\ &= (2\,x - 3\,y)(4\,x^2 + 6\,xy + 9\,y^2). \end{aligned}$$

82. *The sum of two cubes.*

E.g. $$\begin{aligned} 27\,x^3 + 64\,y^3 &= (3\,x + 4\,y)[(3\,x)^2 - (3\,x)(4\,y) + (4\,y)^2] \\ &= (3\,x + 4\,y)(9\,x^2 - 12\,xy + 16\,y^2). \end{aligned}$$

FACTORS OF TRINOMIALS

83. *Trinomial squares.*

E.g. $$a^2 + 2\,ab + b^2 = (a + b)^2 = (a + b)(a + b),$$

and $$a^2 - 2\,ab + b^2 = (a - b)^2 = (a - b)(a - b).$$

84. *Trinomials of the form $x^2 + px + q$.*

E.g. $$x^2 + 3x - 10 = (x+5)(x-2).$$

A trinomial of this form has two binomial factors, $x + a$ and $x + b$, if two numbers a and b can be found whose product is q and whose algebraic sum is p.

85. *Trinomials of the form $mx^2 + nx + r$.*

E.g. $$6x^2 + 7x - 20 = (3x-4)(2x+5).$$

A trinomial of this form has two binomial factors of the type $ax + b$ and $cx + d$, if four numbers, a, b, c, d, can be found such that $ac = m$, $bd = r$, and $ad + bc = n$. See § 142, E. C.

86. *Trinomials which reduce to the difference of two squares.*

E.g. $$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2+y^2)^2 - x^2y^2$$
$$= (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

In this case x^2y^2 is both added to and subtracted from the expression, whereby it becomes the difference of two squares. Evidently the term added and subtracted must itself be a *square*, and hence the degree of the trinomial must be 4 or a multiple of 4, since the degree of the middle term is *half* that of the trinomial.

Ex. $$4a^8 - 16a^4b^4 + 9b^8 = 4a^8 - 12a^4b^4 + 9b^8 - 4a^4b^4$$
$$= (2a^4 - 3b^4)^2 - 4a^4b^4$$
$$= (2a^4 - 3b^4 + 2a^2b^2)(2a^4 - 3b^4 - 2a^2b^2).$$

EXERCISES ON BINOMIALS AND TRINOMIALS

Factor the following:

1. $a^3 + b^3$.
2. $a^3 - b^3$.
3. $(a+b)^3 - c^3$.
4. $(a+b)^3 + c^3$.
5. $7ax^2 - 56a^4x^5$.
6. $a^5 - ab^4$.
7. $121x^7 - 4xy^4$.
8. $\frac{1}{8}a^3 + \frac{1}{125}b^3$.
9. $\frac{1}{8}r^3 - \frac{9}{32}rs^2$.
10. $8r^4 - 27r$.
11. $(a+b)^2 - c^2$.
12. $c^2 - (a-b)^2$.
13. $5c^2 + 7cd - 6d^2$.
14. $x^4 - 3x^2y^2 + y^4$.
15. $4x^2 - 12xy + 9y^2$.
16. $x^2 + 11xz + 30z^2$.

17. $6x^2-5xy-6y^2$.

18. $3a^2x^2y^4-69a^2xy^2+336a^2$.

19. $20a^2b^2+23abx-21x^2$.

20. $a^4+2a^2b^2+9b^4$.

21. $48a^8x^4y-75ay^5$.

22. $16a^4x^3y+54ay^4$.

23. $x^4y^2+2x^2yz+z^2$.

24. $a^2+10a-39$.

25. $8a^2y^3-48a^2y^2z+72a^2yz^2$.

26. $4m^8-60m^4n^4+81n^8$.

27. $35a^{2k}-6a^kb^k-9b^{2k}$.

28. $(a+b)^2-(c-d)^2$.

29. $72a^2x^2-19axy^2-40y^4$.

30. $4(a-3)^6-37b^2(a-3)^3+9b^4$.

31. $6(x+y)^2+5(x^2-y^2)-6(x-y)^2$.

32. $9(x-a)^2-24(x-a)(x+a)+16(x+a)^2$.

33. $12(c+d)^2-7(c+d)(c-d)-12(c-d)^2$.

34. $(a^2+5a-3)^2-25(a^2+5a-3)+150$.

FACTORS OF POLYNOMIALS OF FOUR TERMS

A polynomial of four terms may be readily factored in case it is in any one of the forms given in the next three paragraphs.

87. *It may be the cube of a binomial.*

Ex. 1. $a^3-3a^2b+3ab^2-b^3=(a-b)^3$.

Ex. 2. $8x^3+36x^2y+54xy^2+27y^3$

$$=(2x)^3+3(2x)^2(3y)+3(2x)(3y)^2+(3y)^3$$

$$=(2x+3y)^3.$$ See Ex. 34, (d), p. 23.

88. *It may be resolvable into the difference of two squares.*

In this case three of the terms must form a trinomial square.

Ex. 1. $a^2-c^2+2ab+b^2=(a^2+2ab+b^2)-c^2$

$$=(a+b)^2-c^2=(a+b+c)(a+b-c).$$

Ex. 2. $4x^2+z^6-4x^4-1=z^6-4x^4+4x^2-1$

$$=z^6-(4x^4-4x^2+1)=z^6-(2x^2-1)^2$$

$$=(z^3+2x^2-1)(z^3-2x^2+1).$$

89. *A binomial factor may be shown by grouping the terms.*

In this case the terms are grouped by twos as in the following examples.

Ex. 1. $ax + ay + bx^2 + bxy = (ax + ay) + (bx^2 + bxy)$
$= a(x + y) + bx(x + y) = (a + bx)(x + y).$

Ex. 2. $ax + bx + a^2 - b^2 = (ax + bx) + (a^2 - b^2)$
$= x(a + b) + (a - b)(a + b) = (x + a - b)(a + b).$

EXERCISES

Factor the following polynomials:

1. $x^3 + 3x^2y + 3xy^2 + y^3.$
2. $8a^3 - 36a^2b + 54ab^2 - 27b^3.$
3. $4a^4 - 4a^2b^2 + b^4 - 16x^2.$
4. $2ad + 3bc + 2ac + 3bd.$
5. $27x^3 - 54x^2y + 36xy^2 - 8y^3.$
6. $36a^4 - 24a^3 + 24a - 16.$
7. $mnx^2 - mrx - rn^2x + r^2n.$
8. $a^2b^2 - a^2bc^2n - abn + an^2c^2.$
9. $2y^2 + 4by + 3cy + 6bc.$
10. $bcyz + c^2z^2 + bdy + dcz.$
11. $5a^2c + 12cd - 6ad - 10ac^2.$
12. $a^2 - b^2x^2 + acx^2 - bcx^3.$
13. $b^3c^2 - c^2y^3 - b^3y^2 + y^5.$
14. $a^{3k} - 2a^{2k}b^k - 2a^kb^{2k} + b^{3k}.$
15. $m^{a+b} + m^an^a + m^bn^b + n^{a+b}.$
16. $b^2y^3 - b(c - d)y^2 + d(by - c) + d^2.$

FACTORS FOUND BY GROUPING

90. The discovery of factors by the proper grouping of terms as in § 89 is of wide application. Polynomials of five, six, or more terms may frequently be thus resolved into factors.

Ex. 1. $a^2 + 2ab + b^2 + 5a + 5b = (a + b)^2 + (5a + 5b)$
$= (a + b)(a + b) + 5(a + b) = (a + b + 5)(a + b).$

Ex. 2. $x^2 - 7x + 6 - ax + 6a = x^2 - 7x + 6 - (ax - 6a)$
$= (x - 1)(x - 6) - a(x - 6) = (x - 6)(x - 1 - a).$

Ex. 3. $a^2 - 2ab + b^2 - x^2 - 2xy - y^2$
$= (a^2 - 2ab + b^2) - (x^2 + 2xy + y^2)$
$= (a - b)^2 - (x - y)^2 = (a - b + x - y)(a - b - x + y).$

Ex. 4. $ax^2 + ax - 6a + x^2 + 7x + 12$

$$= a(x^2 + x - 6) + (x^2 + 7x + 12)$$
$$= a(x + 3)(x - 2) + (x + 3)(x + 4)$$
$$= (x + 3)[a(x - 2) + x + 4] = (x + 3)(ax - 2a + x + 4).$$

In some cases the grouping is effective only after a term has been separated into two parts.

Ex. 5. $2a^3 + 3a^2 + 3a + 1 = a^3 + (a^3 + 3a^2 + 3a + 1)$

$$= a^3 + (a + 1)^3 = (a + a + 1)[a^2 - a(a + 1) + (a + 1)^2]$$
$$= (2a + 1)(a^2 + a + 1).$$

As soon as the term $2a^3$ is separated into two terms the expression is shown to be the sum of two cubes.

Again, the grouping may be effective after a term has been both added and subtracted:

Ex. 6. $a^4 + b^4 = (a^4 + 2a^2b^2 + b^4) - 2a^2b^2$

$$= (a^2 + b^2)^2 - (ab\sqrt{2})^2$$
$$= (a^2 + b^2 + ab\sqrt{2})(a^2 + b^2 - ab\sqrt{2}).$$

In this case the factors are irrational as to one coefficient. Such factors are often useful in higher mathematical work.

EXERCISES

Factor the following:

1. $x^2 - 2xy + y^2 - ax + ay.$
2. $a^2 - ab + b^2 + a^3 + b^3.$
3. $a^3 - b^3 - a^2 - ab - b^2.$
4. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2.$
5. $a^4 + 2a^3b - a^2c^2 + a^2b^2 - 2abc^2 - b^2c^2.$
6. $x^4 - y^4 + ax^2 + ay^2 - x^2 - y^2.$
7. $a^4 + a^2b^2 + b^4 + a^3 + b^3.$

In 7 group the first three and the last two terms.

8. $a^3 - 1 + 3x - 3x^2 + x^3.$ Group the last four terms.
9. $x^3 + x^2 + 3x + y^3 - y^2 + 3y.$

Group in pairs, the 1st and 4th, 2d and 5th, 3d and 6th terms.

10. $x^4 + x^3y - xy^3 - y^4 + x^3 - y^3.$
11. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 - x^4.$

12. $x^4+4x^2z-4y^2+4yw+4z^2-w^2$.

13. $2a^2-12b^2+3bd-5ab-9bc-6ac+2ad$.

Group the terms: $2a^2-5ab-12b^2$.

14. $a^2+ab-4ac-2b^2+4bc+3ad-3bd$.

15. a^3+2-3a. Group thus: $(a^3-1)+(3-3a)$.

16. $4a^2+a-8ax-x+4x^2$.

17. $3a^2-8ab+4b^2+2ac-4bc$.

18. x^8+y^8, also $x^{16}+y^{16}$.

19. $a^6+2a^3b^3+b^6-2a^4b-2ab^4$.

20. a^3-3a^2+4. Group thus: $(a^3-2a^2)+(4-a^2)$.

21. $a^2c-ac^2-a^2b+ab^2-b^2c+bc^2$.

22. $a^2b-a^2c+b^2c-ab^2+ac^2-bc^2$.

23. $3x^3-x^2-4x+2$. Add and subtract $-2x^2$.

24. $2x^3-11x^2+18x-9$. Add and subtract $9x^2$.

FACTORS FOUND BY THE FACTOR THEOREM

91. It is possible to determine in advance whether a polynomial in x is divisible by a binomial of the form $x-a$.

E.g. In dividing $x^4-4x^3+7x^2-7x+2$ by $x-2$, the quotient is found to be x^3-2x^2+3x-1.

Since *Quotient* $\times$ *Divisor* $\equiv$ *Dividend*, we have

$$(x-2)(x^3-2x^2+3x-1)\equiv x^4-4x^3+7x^2-7x+2.$$

As this is an *identity*, it holds for all values of x. For $x=2$ the factor $(x-2)$ is zero, and hence the left member is zero, § 22.

Hence for $x=2$ the right member must also be zero. This is indeed the case, viz.:

$$2^4-4\cdot 2^3+7\cdot 2^2-7\cdot 2+2=16-32+28-14+2=0.$$

Hence if $x-2$ is a factor of $x^4-4x^3+7x^2-7x+2$, the latter must reduce to zero for $x=2$.

92. In general let D represent any polynomial in x. Suppose D has been divided by $x-a$ until the remainder no longer contains x. Then, calling the quotient Q and the remainder R, we have the identity

$$D \equiv Q(x-a)+R, \tag{1}$$

which holds for all values of x.

The substitution of a for x in (1) does not affect R, reduces $Q(x-a)$ to zero, and may or may not reduce D to zero.

(1) If $x=a$ reduces D to zero, then $0 \equiv 0+R$. Hence R is zero, and the division is exact. That is, $x-a$ is a factor of D.

(2) If $x=a$ does *not* reduce D to zero, then R is not zero, and the division is not exact. That is, $x-a$ is not a factor of D.

Hence: *If a polynomial in x reduces to zero when a particular number a is substituted for x, then $x-a$ is a factor of the polynomial, and if the substitution of a for x does not reduce the polynomial to zero, then $x-a$ is not a factor.*

This principle is often called the **factor theorem**.

93. In applying the factor theorem the trial divisor must always be written in the form $x-a$.

Ex. 1. Factor $x^4+6x^3+3x^2+x+3$.

If there is a factor of the form $x-a$, then the only possible values of a are the various divisors of 3, namely $+1$, -1, $+3$, -3.

To test the factor $x+1$, we write it in the form $x-(-1)$ where $a=-1$. Substituting -1 for x in the polynomial, we have

$$1-6+3-1+3=0.$$

Hence $x+1$ is a factor.

On substituting $+1$, $+3$, -3 for x successively, no one reduces the polynomial to zero. Hence $x-1$, $x-3$, $x+3$ are not factors.

Ex. 2. Factor $3x^3-x^2-4x+2$.

If $x-a$ is a factor, then a must be a factor of $+2$. We therefore substitute, $+2$, -2, $+1$, -1 and find the expression becomes zero when $+1$ is substituted for x. Hence $x-1$ is a factor. The other factor is found by division to be $3x^2+2x-2$, which is prime.

Hence $$3x^3-x^2-4x+2=(x-1)(3x^2+2x-2).$$

EXERCISES

Factor by means of the factor theorem:

1. $3x^3-2x^2+5x-6$.
2. $2x^3+3x^2-3x-4$.
3. $2x^3+x^2-12x+9$.
4. $x^3+9x^2+10x+2$.
5. a^3-3a+2.
6. m^3+5m^2+7m+3.
7. $x^4+3x^3-3x^2-7x+6$.
8. $3r^3+5r^2-7r-1$.
9. $2z^3+7z^2+4z+3$.
10. $a^3-6a^2+11a-6$.

11. Show by the factor theorem that x^k-a^k contains the factor $x-a$ if k is *any* integer.

12. Show that x^k-a^k contains the factor $x+a$ if k is any *even* integer.

13. Show that x^k+a^k contains the factor $x+a$ if k is any *odd* integer.

14. Show that x^k+a^k contains neither $x+a$ nor $x-a$ as a factor if k is an *even* integer.

MISCELLANEOUS EXERCISES ON FACTORING

1. $20a^3x^3y-45a^3xy^3$.
2. $24am^5n^2-375am^2n^5$.
3. $432ar^4s+54ars^4$.
4. $16x^2-72xy+81y^2$.
5. $162a^3b+252a^2b^2+98ab^3$.
6. $48x^5y-12x^3y-12x^2y+3y$.
7. $12a^2bx^2+8ab^2x^2+18a^2bxy+12ab^2xy$.
8. $18x^3y-39x^2y^2+18xy^3$.
9. $4x^2-9xy+6x-9y+4x+6$.
10. $6x^2-13xy+6y^2-3x+2y$.
11. $6x^4+15x^2y^2+9y^4$.
12. $16x^4+24x^2y^2+8y^4$.
13. $15x^4+24x^2y^2+9y^4$.
14. a^6+y^6.
15. $a^{12}+y^{12}$.
16. a^8-y^8.
17. $a^{16}-y^{16}$.
18. $a^8+a^4y^4+y^8$.
19. a^3+a-2.
20. $a^8-18a^4y^4+y^8$.
21. $a^{16}-6a^8y^8+y^{16}$.
22. x^3+4x^2+2x-1.
23. $3x^3+2x^2-7x+2$.

24. $a^8-3a^4y^4+y^8$.

25. a^3+a^2+a+1.

26. $a^3+9a^2+16a+4$.

27. $2x^4+x^3y+2x^2y^2+xy^3$.

28. $m^5+m^4a+m^3a^2+m^2a^3+ma^4+a^5$.

29. $(x-2)^3-(y-z)^3$.

30. $a^6+b^6+2ab(a^4-a^2b^2+b^4)$.

31. $x^5y^5+x^4y^4z+x^3y^3z^2+x^2y^2z^3+xyz^4+z^5$.

32. $8a^3+6ab(2a-3b)-27b^3$.

33. $a(x^3+y^3)-ax(x^2-y^2)-y^2(x+y)$.

34. $a^3-b^3+3b^2c-3bc^2+c^3$.

35. $a^4+2a^3b-2ab^2c-b^2c^2$.

36. $a^4+2a^3b+a^2b^2-a^4b^2-2a^2b^2c-b^2c^2$.

SOLUTION OF EQUATIONS BY FACTORING

94. Many equations of higher degree than the first may be solved by factoring. (See §§ 144–146, E. C.)

Ex. 1. Solve $$2x^3-x^2-5x-2=0. \quad (1)$$

Factoring the left member of the equation, we have

$$(x-2)(x+1)(2x+1)=0. \quad (2)$$

A value of x which makes one factor zero makes the whole left member zero and so satisfies the equation. Hence $x=2$, $x=-1$, $x=-\frac{1}{2}$ are roots of the equation.

To solve an equation by this method first reduce it to the form $A=0$, and then factor the left member. Put each factor equal to zero and solve for x. The results thus obtained are roots of the original equation.

Ex. 2. Solve $$x^3-12x^2=12-35x. \quad (1)$$

Transposing and factoring, $(x-4)(x^2-8x+3)=0$. (2)

Hence the roots of (1) are the roots of $x-4=0$ and $x^2-8x+3=0$. From $x-4=0$, $x=4$. The quadratic expression x^2-8x+3 cannot be resolved into *rational* factors. See § 155.

EXERCISES

Solve each of the following equations by factoring, obtaining all roots which can be found by means of rational factors.

1. $x^3 + 3x^2 = 28x$.
2. $6x^3 + 8x + 5 = 19x^2$.
3. $x^4 + 12x^2 + 3 = 7x^3 + 9x$.
4. $x^3 = -2x^2 + 5x + 6$.
5. $x^3 - 4x^2 = 4x + 5$.
6. $2x^3 + 3x = 9x^2 - 14$.
7. $5x^3 + x^2 - 14x + 8 = 0$.
8. $2x^3 + x^2 = 14x - 3$.
9. $x^4 + 2x^3 + 12 = 7x^2 + 8x$.
10. $x^4 + x + 6 = x^3 + 7x^2$.

COMMON FACTORS AND MULTIPLES

95. If each of two or more expressions is resolved into prime factors, then their **Highest Common Factor** (H. C. F.) is at once evident as in the following example. See § 182, E. C.

Given (1) $x^4 - y^4 = (x^2 + y^2)(x + y)(x - y)$,

(2) $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

Then $(x + y)(x - y) = x^2 - y^2$ is the H. C. F. of (1) and (2).

In case only one of the given expressions can be factored by inspection, it is usually possible to select those of its factors, if any, which will divide the other expressions and so to determine the H. C. F.

Ex. Find the H. C. F. of $6x^3 + 4x^2 - 3x - 2$,
and $2x^4 + 2x^3 + x^2 - x - 1$.

By grouping we find:

$$6x^3 + 4x^2 - 3x - 2 = 2x^2(3x + 2) - (3x + 2)$$
$$= (2x^2 - 1)(3x + 2).$$

The other expression cannot readily be factored by any of the methods thus far studied. However, if there is a common factor, it must be either $2x^2 - 1$ or $3x + 2$. We see at once that it cannot be $3x + 2$. (Why?) By actual division $2x^2 - 1$ is found to be a factor of $2x^4 + 2x^3 + x^2 - x - 1$. Hence $2x^2 - 1$ is the H. C. F.

96. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is readily found if these are resolved into prime factors. See § 185, E. C.

Ex. 1. Given $6abx - 6aby = 2 \cdot 3ab(x - y),$ (1)

$$8a^2x + 8a^2y = 2^3a^2(x + y), \quad (2)$$

$$36b^3(x^2 - y^2)(x + y) = 2^23^2b^3(x - y)(x + y)^2. \quad (3)$$

The L. C. M. is $2^3 \cdot 3^2 a^2b^3(x - y)(x + y)^2$, since this contains all the factors of (1), all the factors of (2) not found in (1), and all the factors of (3) not found in (1) and (2), with no factors to spare.

In case only one of the given expressions can be factored by inspection, it may be found by actual division whether or not any of these factors will divide the other expressions.

Ex. 2. Find the L. C. M. of $6x^3 - x^2 + 4x + 3,$ (1)

and $6x^3 + 3x^2 - 10x - 5.$ (2)

(1) is not readily factored. Grouping by twos, the factors of (2) are $3x^2 - 5$ and $2x + 1$. Now $3x^2 - 5$ is not a factor of (1). (Why?) Dividing (1) by $2x + 1$ the quotient is $3x^2 - 2x + 3$.

Hence $6x^3 - x^2 + 4x + 3 = (2x + 1)(3x^2 - 2x + 3),$

$$6x^3 + 3x^2 - 10x - 5 = (2x + 1)(3x^2 - 5).$$

Hence the L. C. M. is $(2x + 1)(3x^2 - 2x + 3)(3x^2 - 5)$.

Ex. 3. Find the L. C. M. of $a^3 + 2a^2 - a - 2,$ (1)

and $10a^3 - 3a^2 + 4a + 1.$ (2)

By means of the factor theorem, $a - 1$, $a + 1$, and $a + 2$ are found to be factors of (1), but none of the numbers, 1, − 1, − 2, when substituted for a in (2) will reduce it to zero. Hence (1) and (2) have no factors in common. The L. C. M. is therefore the product of the two expressions; viz. $(a + 1)(a - 1)(a + 2)(10a^3 - 3a^2 + 4a + 1)$.

97. The H. C. F. of three expressions may be obtained by finding the H. C. F. of two, and then the H. C. F. of this result and the third expression. Similarly the L. C. M. of three expressions may be obtained by finding the L. C. M. of two of them, and then the L. C. M. of this result and the third expression.

This may be extended to any number of expressions.

EXERCISES

Find the H.C.F. and also the L.C.M. in each of the following:-

1. $x^2+y^2,\ x^6+y^6$.
2. $x^2+xy+y^2,\ x^3-y^3$.
3. $x^2-5x-6,\ x^2-2x-3,\ x^2+19x+18$.
4. $x^4-6x^2+1,\ x^3+x^2-3x+1,\ x^3+3x^2+x-1$.
5. $162a^3b+252a^2b^2+9ab^3,\ 54a^3+42a^2b$.
6. $2x^3+x^2-8x+3,\ x^2+2x-1$.
7. $3r^3+5r^2-7r-1,\ 3r^2+8r+1$.
8. $a^3-3a^2+4,\ ax-ab-2x+2b$.
9. $a^6+2a^3b^3+b^6-2a^4b-2ab^4,\ a^3-2ab+b^3$.
10. $8a^3-36a^2b+54ab^2-27b^3,\ 4a^2-9b^2$.
11. $36a^4-9a^2-24a-16,\ 12a^3-6a^2-8a$.
12. $2y^2+4by+3cy+6bc,\ y^2-3by-10b^2$
13. $x^{16}-y^{16},\ x^8-y^8,\ x^4-y^4$.
14. $m^3+8m^2+7m,\ m^3+3m^2-m-3,\ m^3-7m-6$.

98. An **important principle** relating to common factors is illustrated by the following example:

Given $\qquad x^2+7x+10=(x+5)(x+2), \qquad (1)$

and $\qquad x^2-x-6=(x-3)(x+2). \qquad (2)$

Add (1) and (2), $\qquad 2x^2+6x+4=2(x+1)(x+2). \qquad (3)$

Subtract (2) from (1), $\qquad 8x+16=8(x+2). \qquad (4)$

We observe that $x+2$, which is a common factor of (1) and (2), is also a factor of their sum (3), and of their difference (4). This example is a special case of the following principle.

99. *A common factor of two expressions is also a factor of the sum or difference of any multiples of those expressions.*

For let A and B be any two expressions having the common factor f. Then if k and l are the remaining factors of A and B respectively,

$$A = fk \text{ and } B = fl.$$

Also let mA and nB be any multiples of A and B.

Then $mA = mfk$ and $nB = nfl$, from which we have:

$$mA \pm nB = mfk \pm nfl = f(mk \pm nl).$$

Hence f is a factor of $mA \pm nB$.

100. Another important principle is the following: *If* f *is a factor of* $mA \pm nB$ *and also of* A, *then* f *is a factor of* B, *provided* n *has no factor in common with* A.

For let f be a factor of $mA \pm nB$ and also of A, where mA and nB are integral multiples of the expressions A and B.

Then f must divide both mA and nB. We know it divides mA because it was given as a factor of A. If it divides nB, it must be a factor of either n or B. It cannot be a factor of n, for then A and n would have a factor in common contrary to agreement. Hence f is a factor of B.

101. By successive applications of the above principles it is possible to find the H.C.F. of any two integral expressions.

Ex. 1. Find the H.C.F. of $9x^4 - x^2 + 2x - 1$, (1)

and $27x^5 + 8x^2 - 3x + 1$. (2)

Multiplying (1) by $3x$ and subtracting from (2), we have

$$\begin{array}{r} 27x^5 + 8x^2 - 3x + 1 \\ \underline{27x^5 - 3x^3 + 6x^2 - 3x} \\ 3x^3 + 2x^2 + 1 \end{array} \qquad (3)$$

By § 99, any common factor of (1) and (2) is a factor of (3).

Calling expressions (1) and (2) B and A respectively of principle **2**, then (3) is $A - 3x \cdot B$; and since the multiplier, $3x$, has no factor in common with (2), it follows from the principle that any common factor of (3) and (2) is a factor of (1), and also that any common factor of (3) and (1) is a factor of (2). Hence (1) and (3) have the *same common factors*, that is, the same H. C. F. as (1) and (2). Therefore we proceed to obtain the H. C. F. of

$$9x^4 - x^2 + 2x - 1, \tag{1}$$

and

$$3x^3 + 2x^2 + 1. \tag{3}$$

Multiplying (3) by $3x$ and subtracting from (1) we have

$$-6x^3 - x^2 - x - 1. \tag{4}$$

By argument similar to that above, (3) and (4) have the same H. C. F. as (1) and (3) and hence the same as (1) and (2). Multiplying (3) by 2 and adding to (4) we have,

$$3x^2 - x + 1. \tag{5}$$

Then the H. C. F. of (5) and (3) is the same as that of (1) and (2). Multiplying (5) by x and subtracting from (3), we have

$$3x^2 - x + 1. \tag{6}$$

Then the H. C. F. of (5) and (6) is the same as that of (1) and (2). But (5) and (6) are identical, that is, their H. C. F. is $3x^2 - x + 1$. Hence this is the H. C. F. of (1) and (2).

The work may be conveniently arranged thus:

$$\begin{array}{lrcrl}
(1) & 9x^4 \qquad - x^2 + 2x - 1 & \qquad & 27x^5 \qquad + 8x^2 - 3x + 1 & (2) \\
 & \underline{9x^4 + 6x^3 \qquad + 3x} & & \underline{27x^5 - 3x^3 + 6x^2 - 3x} & \\
(4) & -6x^3 - x^2 - x - 1 & & 3x^3 + 2x^2 \qquad + 1 & (3) \\
 & \underline{6x^3 + 4x^2 \qquad + 2} & & \underline{3x^3 - x^2 + x} & \\
(5) & 3x^2 - x + 1 & & 3x^2 - x + 1 & (6)
\end{array}$$

The object at each step is to obtain a new expression of as low a degree as possible. For this purpose the highest powers are eliminated step by step by the method of addition or subtraction.

E.g. In Ex. 1, x^5 was eliminated first, then x^4, and then x^3.

By principles **1** and **2**, each new expression contains all the factors common to the given expressions. Hence, whenever an expression is reached which is *identical with the preceding one*, this is the H. C. F.

102. The process is further illustrated as follows:

Ex. 2. Find the H. C. F. of $2x^3-2x^2-3x-2$,

and $3x^3-x^2-2x-16$.

Arranging the work as in Ex. 1, we have

(1)	$2x^3-2x^2-3x-2$	$3x^3-x^2-2x-16$	(2)
	$4x^3-4x^2-6x-4$	$6x^3-2x^2-4x-32$	
	$4x^3+5x^2-26x$	$6x^3-6x^2-9x-6$	
(4)	$-9x^2+20x-4$	$4x^2+5x-26$	(3)
	$9x^2-18x$	$36x^2+45x-234$	
(7)	$2x-4$	$-36x^2+80x-16$	
(8)	$x-2$	$125x-250$	(5)
		$x-2$	(6)

Explanation. To eliminate x^3, we multiply (1) by 3 and (2) by 2 and subtract, obtaining (3).

To eliminate x^2 from (3), we need another expression of the second degree. To obtain this we multiply (1) by 2 and (3) by x and subtract, obtaining (4).

Using (4) and (3), we eliminate x^2, obtaining (5). Since (5) contains all factors common to (1) and (2), and since 125 is not such a factor, this is discarded without affecting the H. C. F., giving (6).

Multiplying (6) by 9 and adding to (4) we have (7). Discarding the factor 2 gives (8) which is identical with (6). Hence $x-2$ is the H. C. F. sought.

103. Any *monomial* factors should be removed from each expression at the outset. If there are such factors *common* to the given expressions, these form a part of the H. C. F.

When this is done, then any monomial factor of any one of the derived expressions may be at once discarded without affecting the H. C. F., as in (5) of the preceding example.

In this way also the hypothesis of principle **2** is always fulfilled; namely, that at every step the multiplier of one expression shall have no factor in common with the other expression.

Ex. 3. Find the H. C. F. of $3x^3 - 7x^2 + 3x - 2$,

and $x^4 - x^3 - x^2 - x - 2$.

$$
\begin{array}{lrlr}
(1) & 3x^3 - 7x^2 + 3x - 2 & x^4 - x^3 - x^2 - x - 2 & (2) \\
 & 12x^3 - 28x^2 + 12x - 8 & 3x^4 - 3x^3 - 3x^2 - 3x - 6 & \\
 & \underline{12x^3 - 18x^2 - 3x - 18} & \underline{3x^4 - 7x^3 + 3x^2 - 2x } & \\
(4) & -10x^2 + 15x + 10 & 4x^3 - 6x^2 - x - 6 & (3) \\
(5) & -5(2x+1)(x-2). & &
\end{array}
$$

Explanation. To eliminate x^4, we multiply (1) by x and (2) by 3 and subtract, obtaining (3).

To eliminate x^3, we multiply (1) by 4 and (3) by 3 and subtract, obtaining (4).

At this point the work may be shortened by factoring (4) as in (5). We may now reject, not only the factor -5, but also $2x + 1$, which is a factor of neither (1) nor (2), since $2x$ does not divide the highest power of either expression. But $x - 2$ is seen to be a factor of (2), by §§ 91, 92, and hence it is a common factor of (2) and (4) and therefore of (1) and (2). Hence $x - 2$ is the H. C. F. sought.

EXERCISES

Find the H.C.F. of the following pairs of expressions:

1. $a^3 + 6a^2 + 6a + 5$, $a^3 + 4a^2 - 4a + 5$.
2. $x^4 - 2x^3 - 2x^2 + 5x - 2$, $x^4 - 4x^3 + 6x^2 - 5x + 2$.
3. $2x^3 - 9x^2 - 13x - 4$, $x^3 - 12x^2 + 31x + 28$.
4. $x^4 - 5x^2 + 3x - 2$, $x^4 - 3x^3 + 3x^2 - 3x + 2$.
5. $2x^3 - 9x^2 + 8x - 2$, $2x^3 + 5x^2 - 5x + 1$.
6. $3a^4 - 2a^3 + 10a^2 - 6a + 3$, $2a^4 + 3a^3 + 5a^2 + 9a - 3$.
7. $15x^4 + 19x^3 - 44x^2 - 15x + 9$,
   $15x^4 - 6x^3 + 51x^2 + 11x - 15$.
8. $r^5 + 2r^4 - 2r^3 - 8r^2 - 7r - 2$, $r^5 - 2r^4 - 2r^3 + 4r^2 + r - 2$.

104. The following principle enables us to find the L. C. M. of two expressions by means of the method which has just been used for finding the H. C. F.

The L. C. M. of two expressions is equal to the product of either expression and the quotient obtained by dividing the other by the H. C. F. of the two expressions.

For let A and B be two expressions whose H. C. F. is F so that $A = mF$ and $B = nF$. Hence the L. C. M. of A and B is mnF. But $mnF = m \cdot nF = mB$. Also $mnF = n \cdot mF = nA$. Therefore the L. C. M. is either mB or nA, where $m = A \div F$ and $n = B \div F$.

Ex. Find the L. C. M. of $9x^4 - x^2 + 2x - 1$, (1)

and $27x^5 + 8x^2 - 3x + 1$. (2)

The H. C. F. was found in § 101 to be $3x^2 - x + 1$.
Dividing (1) by $3x^2 - x + 1$ we have $3x^2 + x - 1$.

Hence the L. C. M. of (1) and (2) is

$$(27x^5 + 8x^2 - 3x + 1)(3x^2 + x - 1).$$

EXERCISES

Find the L. C. M. of each of the following sets.

1. $a^4 + a^3 + 2a^2 - a + 3,\ a^4 + 2a^3 + 2a^2 - a + 4.$
2. $a^3 - 6a^2 + 11a - 6,\ a^3 - 9a^2 + 26a - 24.$
3. $2a^3 + 3a^2b - 2ab^2 - 3b^3,\ 2a^4 - a^3b - 2a^2b^2 + 4ab^3 - 3b^4.$
4. $2a^3 - a^2b - 13ab^2 - 6b^3,$
   $2a^4 - 5a^3b - 11a^2b^2 + 20ab^3 + 12b^4.$
5. $4a^3 - 15a^2 - 5a - 3,\ 8a^4 - 34a^3 + 5a^2 - a + 3,$
   $2a^3 - 7a^2 + 11a - 4.$
6. $a^4 + a^2 + 1,\ a^3 + 2a^2 - 2a + 3.$
7. $2k^3 - k^2l - 13kl^2 + 5l^3,\ 3k^3 - 16k^2l + 24kl^2 - 7l^3.$
8. $12r^4 - 20r^3s - 15r^2s^2 + 35rs^3 - 12s^4,$
   $6r^3 - 7r^2s - 11rs^2 + 12s^3.$
9. $2a^3 - 7a^2 + 6a - 2,\ a^3 + 2a^2 - 13a + 10,\ a^3 + 6a^2 + 6a + 5.$
10. $x^3 - xy^2 + yx^2 - y^3,\ 2x^3 + x^2y + xy^2 + 2y^3,$
    $2x^3 + 3x^2y + 3xy^2 + 2y^3.$

CHAPTER VI

POWERS AND ROOTS

105. Each of the operations thus far studied leads to a **single result.**

E.g. Two numbers have one and only one *sum*, § 2, and one and only one *product*, § 7.

When a number is subtracted from a given number, there is one and only one *remainder*, § 6.

When a number is divided by a given number, there is one and only one *quotient*, § 11.

We are now to study an operation which leads to **more than one result**; namely, the operation of finding roots.

Thus both 3 and -3 are square roots of 9, since $3 \cdot 3 = 9$, and also $(-3)(-3) = 9$; this is often indicated by $\sqrt{9} = \pm 3$. See § 114.

106. The operations of addition, subtraction, multiplication, and division are **possible** in all cases *except dividing by zero*, which is explicitly ruled out, §§ 24, 25.

Division is possible in general because *fractions* are admitted to the number system, and subtraction is possible in general because *negative numbers* are admitted. Thus $7 \div 3 = 2\frac{1}{3}$, $5 - 7 = -2$.

107. The operation of finding roots is not possible in all cases, unless other numbers besides positive and negative integers and fractions are admitted to the number system.

E.g. The number $\sqrt{2}$ is not an *integer* since $1^2 = 1$ and $2^2 = 4$.

Suppose $\sqrt{2} = \frac{a}{b}$ a fraction reduced to its lowest terms, so that a and b have no common factor. Then $\frac{a^2}{b^2} = 2$. But this is impossible, for if b^2 exactly divides a^2, then a and b must have factors in common. Hence $\sqrt{2}$ is not a *fraction*.

108. If a positive number is not the square of an integer or a fraction, a number may be found in terms of integers and fractions whose square differs from the given number by as little as we please. See p. 228, E. C.

E.g. 1.41, 1.414, 1.4141 are successive numbers whose squares differ by less and less from 2. In fact $(1.4141)^2$ differs from 2 by less than .0004, and by continuing the process by which these numbers are found, § 170, E. C., a number may be reached whose square differs from 2 by as little as we please.

1.41, 1.414, 1.4141, etc., are successive approximations to the number which we call *the square root of* 2, and which we represent by the symbol, $\sqrt{2}$.

109. Definition. If a number is not the *k*th power of an integer or a fraction, but if its *k*th root can be *approximated* by means of integers and fractions to any specified degree of accuracy, then such a *k*th root is called an **irrational number.** See § 36.

E.g. $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[3]{5}$, etc., are irrational numbers, whereas $\sqrt{4}$, $\sqrt[3]{8}$, are rational numbers.

It is shown in higher algebra that irrational numbers correspond to definite points on the line of the number scale, § 49, E. C., just as integers and fractions do.

We, therefore, now enlarge the number system to include **irrational numbers** as well as integers and fractions.

It will be found also in higher work that there are other kinds of irrational numbers besides those here defined.

The set of numbers consisting of all rational and irrational numbers is called the **real number system.**

110. Even with the number system as thus enlarged, it is still not possible to find roots in all cases. The exception is the **even root of a negative number.**

E.g. $\sqrt{-4}$ is neither $+2$ nor -2, since $(+2)^2 = +4$ and $(-2)^2 = +4$, and no approximation to this root can be found as in the case of $\sqrt{2}$.

111. Definition. The indicated *even* root of a negative number, or any expression containing such a root, is called an **imaginary number**, or more properly, a **complex number.** All other numbers are called **real numbers.**

E.g. $\sqrt{-4}$, $\sqrt[4]{-2}$, $1+\sqrt{-2}$, are *complex numbers,* while 5, $\sqrt[3]{2}$, $1+\sqrt{2}$ are *real numbers.*

Complex numbers cannot be pictured on the line which represents real numbers, but another kind of graphic representation of complex numbers is made in higher algebraic work, and such numbers form the basis of some of the most important investigations in advanced mathematics.

112. With the number system thus enlarged, by the admission of irrational and complex numbers, we have the following **fundamental definition.**

$$(\sqrt[k]{n})^k = n.$$

That is, a kth root of any number n is such a number that, if it be raised to the kth power, the result is n itself.

E.g. $(\sqrt[3]{2})^3 = 2$, $(\sqrt{4})^2 = 4$, $(\sqrt{-2})^2 = -2$.

The imaginary or complex **unit** is $\sqrt{-1}$. By the above definition we have

$$(\sqrt{-1})^2 = -1.$$

In operating upon complex numbers, they should first be expressed in terms of the **imaginary unit.**

E.g. $\sqrt{-2} = \sqrt{2}\cdot\sqrt{-1}$, $\sqrt{-16} = \sqrt{16}\cdot\sqrt{-1} = 4\sqrt{-1}$.

$$\sqrt{-4}\cdot\sqrt{-9} = (\sqrt{4}\cdot\sqrt{-1})(\sqrt{9}\ \sqrt{-1}) = 2\cdot 3(\sqrt{-1})^2 = -6.$$

$$\sqrt{-4}+\sqrt{-9} = \sqrt{4}\cdot\sqrt{-1}+\sqrt{9}\cdot\sqrt{-1} = (2+3)\sqrt{-1} = 5\sqrt{-1}.$$

$$\frac{\sqrt{-16}}{\sqrt{-9}} = \frac{\sqrt{16}\cdot\sqrt{-1}}{\sqrt{9}\cdot\sqrt{-1}} = \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3}.$$

113. By means of irrational and complex numbers it can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc. See § 195, Ex. 17–20.

E.g. The square roots of 9 are $+3$ and -3. The square roots of -9 are $\pm\sqrt{-9} = \pm 3\sqrt{-1}$. The cube roots of 8 are 2, $-1+\sqrt{-3}$ and $-1-\sqrt{-3}$. The fourth roots of 16 are $+2$, -2, $+2\sqrt{-1}$ and $-2\sqrt{-1}$.

Any positive real number has two real roots of *even* degree, one positive and one negative.

E.g. $\sqrt[4]{16} = \pm 2$. The square roots of 3 are $\pm\sqrt{3}$.

Any real number, positive or negative, has one real root of *odd* degree, whose sign is the same as that of the number itself.

E.g. $\sqrt[3]{27} = 3$ and $\sqrt[5]{-32} = -2$.

114. The positive *even* root of a positive real number, or the real *odd* root of any real number, is called the **principal root**.

The positive square root of a negative real number is also sometimes called the *principal* imaginary root.

E.g. 2 is the principal square root of 4, 3 is the principal 4th root of 81; -4 is the principal cube root of -64; and $+\sqrt{-3}$ is the principal square root of -3.

Unless otherwise stated the radical sign is understood to indicate the *principal root.*

The only exception in this book is in such cases as, $\sqrt{4} = \pm 2$, where it represents *either* square root. But in such expressions as $1+\sqrt{2}$, $3 \pm \sqrt[3]{6}$, etc., the *principal root* only is understood.

In all cases it is easily seen from the context in what sense the sign is used.

When it is desired to designate in *particular* the principal root, the symbol $\sqrt{\quad}\rceil$ is used.

E.g. $\sqrt[4]{16}\rceil = 2$, while $\sqrt[4]{16}$ might stand indifferently for 2, -2, $2\sqrt{-1}$, or $-2\sqrt{-1}$.

$\sqrt[3]{8}\rceil = 2$, while $\sqrt[3]{8}$ might represent 2, $-1+\sqrt{-3}$, or $-1-\sqrt{-3}$.

PRINCIPLES INVOLVING POWERS AND ROOTS

115. From § 43, $(2^3)^2 = (2^2)^3 = 2^6 = 64.$

In general, if n and k are any positive integers,

$$(b^k)^n = (b^n)^k = b^{nk}.$$

For $$(b^k)^n = b^k \cdot b^k \cdot b^k \cdots \text{ to } n \text{ factors}$$
$$= b^{k+k+k \cdots \text{ to } n \text{ terms}} = b^{nk}.$$

Likewise, $$(b^n)^k = b^{nk}.$$

Hence: *The nth power of the kth power of any base is the nkth power of that base.*

116. Again, from §§ 43, 44, $(2^3 \cdot 3^2)^2 = 2^6 \cdot 3^4.$
In general, if k, r, and n are any positive integers,

$$(a^k b^r)^n = a^{nk} b^{nr}.$$

For $$(a^k b^r)^n = (a^k b^r) \cdot (a^k b^r) \cdots \text{ to } n \text{ factors}$$
$$= (a^k \cdot a^k \cdots \text{ to } n \text{ factors})(b^k \cdot b^k \cdots \text{ to } n \text{ factors})$$
$$= (a^k)^n \cdot (b^r)^n = a^{nk} b^{nr}.$$

Hence: *The nth power of the product of several factors is the product of the nth powers of those factors.*

117. From § 43, we have $\left(\dfrac{2^3}{3^2}\right)^2 = \dfrac{2^3}{3^2} \cdot \dfrac{2^3}{3^2} = \dfrac{2^6}{3^4}.$

In general, $$\left(\frac{a^k}{b^r}\right)^n = \frac{a^{nk}}{b^{nr}}.$$

For we have $$\left(\frac{a^k}{b^r}\right)^n = \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdots \text{ to } n \text{ factors}$$
$$= \frac{a^{nk}}{b^{nr}}.$$

Hence: *The nth power of the quotient of two numbers equals the quotient of the nth power of those numbers.*

118. It follows from §§ 115, 116, 117, that

Any positive integral power of a monomial is found by multiplying the exponents of the factors by the exponent of the power.

119. We may easily verify that $\sqrt{3^4} = 3^{4 \div 2} = 3^2 = 9.$

In general, if k and r are positive integers and b any positive real number, we have:

$$\sqrt[r]{b^{kr}} = b^{kr \div r} = b^k.$$

For, from § 115, $(b^k)^r = b^{kr}.$

Hence by definition b^k is an rth root of b^{kr}, and since b^k is *real* and *positive*, it is the *principal* rth root of b^{kr} (§ 114).

That is, $\sqrt[r]{b^{kr}} = b^{kr \div r} = b^k.$

Hence: *The principal rth root of the **krth** power of any positive real number is a power of that number whose exponent is* $\boldsymbol{kr \div r = k}$.

E.g. $\sqrt[4]{2^{12}} = 2^{12 \div 4} = 2^3 = 8.$ But it does *not* follow that

$$\sqrt[4]{(-2)^{12}} = (-2)^{12 \div 4} = (-2)^3 = -8,$$

since $(-2)^{12} = (2)^{12}$, and hence $\sqrt[4]{(-2)^{12}} = \sqrt[4]{2^{12}} = +8.$

The corresponding principle holds when b is *negative* if r is *odd* and also when b is *negative* if k is *even.*

E.g. $\sqrt[3]{(-2)^6} = (-2)^{6 \div 3} = (-2)^2 = 4;$ $\sqrt[3]{(-2)^{15}} = (-2)^5 = -32.$

120. Another general principle, if a and b are any real numbers and r any positive integer, is

$$\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$$

For by § 116, $(\sqrt[r]{a} \cdot \sqrt[r]{b})^r = (\sqrt[r]{a})^r \cdot (\sqrt[r]{b})^r,$

And by § 112, $= a \cdot b.$

Hence $ab = (\sqrt[r]{a} \cdot \sqrt[r]{b})^r.$

Taking the principal rth root of both members,

we have $\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$

Hence: *The principal rth root of the product of two positive real numbers equals the product of the principal rth roots of the number.*

When r is *even* the corresponding principle does not hold if a and b are both *negative*.

For example, it is *not true* that $\sqrt{(-4)(-9)} = \sqrt{-4} \cdot \sqrt{-9}$.
For $\sqrt{(-4)(-9)} = \sqrt{36} = 6$; while $\sqrt{-4} \cdot \sqrt{-9}$
$= 2\sqrt{-1} \cdot 3\sqrt{-1} = 6(\sqrt{-1})^2 = -6$. See § 112.

121. Again, if a and b are any positive real numbers and r is any positive integer,

$$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

For we have by §§ 117, 112, $\left(\frac{\sqrt[r]{a}}{\sqrt[r]{b}}\right)^r = \frac{(\sqrt[r]{a})^r}{(\sqrt[r]{b})^r} = \frac{a}{b}$.

Hence, taking the principal rth root of both members,

we have
$$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

That is: *The principal rth root of the quotient of two positive real numbers equals the quotient of the principal rth roots of the numbers.*

E.g. $\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$; $\sqrt[3]{\frac{-8}{27}} = \frac{\sqrt{-8}}{\sqrt[3]{27}} = \frac{-2}{3} = -\frac{2}{3}$.

The corresponding principle does *not* hold when r is *even* if a is positive and b is negative. Thus it is not true that

$$\sqrt{\frac{4}{-9}} = \frac{\sqrt{4}}{\sqrt{-9}} = \frac{2}{3\sqrt{-1}} = \frac{2\sqrt{-1}}{3(\sqrt{-1})^2} = \frac{2\sqrt{-1}}{-3} = -\frac{2}{3}\sqrt{-1}.$$

But we have
$$\sqrt{\frac{4}{-9}} = \sqrt{\frac{4}{9}(-1)} = \frac{2}{3}\sqrt{-1}.$$

If r is odd, the principle holds for *all* real values of a and b.

122. From §§ 119, 120, 121, it follows that:

If a monomial is a perfect power of the kth degree, its kth root may be found by dividing the exponent of each factor by the index of the root.

In applying the above principle to the reduction of algebraic expressions containing letters, it is assumed that the values of the letters are such that the principles apply.

EXERCISES

Find the following indicated powers and roots, and reduce each expression to its simplest form:

1. $(a^3b^4c^5)^7$.
2. $(2^{a+b} \cdot 3^c \cdot 5^b)^{a-b}$.
3. $\left(\dfrac{a^3b^4c^5}{2^3 \cdot 3^2 \cdot 4^3}\right)^2$.
4. $(a^{x-y})^{x^2+xy+y^2}$.
5. $(x^4y^5z^{x+y})^{x-y}$.
6. $\left(\dfrac{5^2b^2mn}{3^7bc^4}\right)^3$.
7. $(3^5 \cdot 4^5 \cdot 2^5)^{a-b}$.
8. $\sqrt[a]{3^{2a} \cdot 2^a \cdot 5^{3a}}$.
9. $\sqrt[3]{\dfrac{-27 \cdot 8a^6}{64c^9d^{6a}}}$.
10. $(a^{m+n-1}b^{m-n}c^{mn})^{m+n}$.
11. $(3^{a+4} \cdot 4^{b-7} \cdot 5^{c-1})^{abx}$.
12. $\sqrt[2a]{3^{6a} \cdot 4^{2a} \cdot 5^{8a} \cdot 7^{4a}}$.
13. $\sqrt[a-b]{3^{a^3-b^3} \cdot 4^{a-b} \cdot 5^{a^2-b^2}}$.
14. $\sqrt{64 \cdot 25 \cdot 256 \cdot 625}$.
15. $\sqrt[3]{27 \cdot 125 \cdot 64 \cdot 3^{6a}}$.
16. $(a-b)^{m-n}(b-c)^{m-n}(a+b)^{m-n}$.
17. $\sqrt{\dfrac{(a-b)^2(a^2+2ab+b^2)}{(a-b)^4(a+b)^2}}$.
18. $\sqrt{\dfrac{(4x^2+4x+1)(4x^2-4x+1)}{36x^4-12x^2+1}}$.
19. $\sqrt[3]{(-343)(-27)x^6(a+b)^{3a}}$.
20. $\sqrt[3]{\dfrac{(-8)(-27)(-125)a^{3m}b^{9n}}{(-1)(-512)(1000)x^{15a}y^{21}}}$.

ROOTS OF POLYNOMIALS

123. In the Elementary Course, pp. 221–224, it was shown that the process for finding the **square root** of a polynomial is obtained by studying the relation of the square, $a^2 + 2\,ab + b^2$, to its square root, $a+b$.

In like manner the process for finding the **cube root** of a polynomial is obtained by studying the relation of the cube, $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ or $a^3 + b(3\,a^2 + 3\,ab + b^2)$, to its cube root, $a + b$.

An example will illustrate the process.

Ex. 1. Find the cube root of

$$27\,m^3 + 108\,m^2n + 144\,mn^2 + 64\,n^3.$$

$$
\begin{array}{rl|l}
\text{Given cube,} & & 27\,m^3+108\,m^2n+144\,mn^2+64\,n^3 \;\big|\; \underline{3\,m+4\,n,\ \text{cube root}} \\
 & a^3 = 27\,m^3 & \underline{\hphantom{108\,m^2n+144\,mn^2+64\,n^3}}\quad \text{1st partial product} \\
 & 3\,a^2 = 27\,m^2 & 108\,m^2n+144\,mn^2+64\,n^3, \quad \text{1st remainder} \\
 & 3\,ab = 36\,mn & \\
 & b^2 = 16\,n^2 & \\
\multicolumn{2}{r|}{3\,a^2+3\,ab+b^2=27\,m^2+36\,mn+16\,n^2} & \underline{108\,m^2n+144\,mn^2+64\,n^3} = b(3\,a^2+3\,ab+b^2) \\
 & & \qquad\qquad\qquad\qquad 0
\end{array}
$$

Explanation. The cube root of the first term, namely $3\,m$, is the first term of the root and corresponds to a of the formula. Cubing $3\,m$ gives $27\,m^3$ which is the a^3 of the formula.

Subtracting $27\,m^3$ leaves $108\,m^2n + 144\,mn^2 + 64\,n^3$, which is the $b(3\,a^2 + 3\,ab + b^2)$ of the formula.

Since b is not yet known, we cannot find completely either factor of $b(3\,a^2 + 3\,ab + b^2)$, but since a has been found, we can get the first term of the factor $3\,a^2+3\,ab+b^2$; viz. $3\,a^2$ or $3(3\,m)^2 = 27\,m^2$, which is the partial divisor. Dividing $108\,m^2n$ by $27\,m^2$ we have $4\,n$, which is the b of the formula.

Then $3\,a^2 + 3\,ab + b^2 = 3(3\,m)^2 + 3(3\,m)(4\,n) + (4\,n)^2 = 27\,m^2 + 36\,mn + 16\,n^2$ is the complete divisor. This expression is then multiplied by $b = 4\,n$, giving $108\,m^2n + 144\,mn^2 + 64\,n^3$, which corresponds to $b(3\,a^2 + 3\,ab + b^2)$ of the formula. On subtracting, the remainder is zero and the process ends. Hence, $3\,m + 4\,n$ is the required root.

Ex. 2. Find the cube root of

$$33x^4 - 9x^5 + x^6 - 63x^3 + 66x^2 - 36x + 8.$$

We first arrange the terms with respect to the exponents of x.

$$\begin{array}{rr|l}
 & & \overline{x^2 - 3x + 2, \text{ cube root}} \\
\text{Given cube,} & & x^6 - 9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8 \\
 & a^3 = x^6 & \\
 & 3a^2 = 3x^4 & -9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8 \\
 & 3a^2 + 3ab + b^2 = 3x^4 - 9x^3 + 9x^2 & -9x^5 + 27x^4 - 27x^3 \\
 & 3a'^2 = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2 & 6x^4 - 36x^3 + 66x^2 - 36x + 8 \\
 & 3a'^2 + 3a'b' + b'^2 = 3x^4 - 18x^3 + 33x^2 - 18x + 4 & 6x^4 - 36x^3 + 66x^2 - 36x + 8 \\
 & & 0
\end{array}$$

The cube root of x^6, or x^2, is the first term of the root. The first partial divisor, which corresponds to $3a^2$ of the formula, is $3(x^2)^2 = 3x^4$. Dividing $-9x^5$ by $3x^4$ we have $-3x$, which is the second term of the quotient, corresponding to b of the formula.

After these two terms of the root have been found, we consider $x^2 - 3x$ as the a of the formula and call it a'. The new partial divisor is $3a'^2 = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2$, and the new b, which we call b', is then found to be 2.

Substituting $x^2 - 3x$ for a' and 2 for b' in $3a'^2 + 3a'b' + b'^2$, we have $3x^4 - 18x^3 + 33x^2 - 18x + 4$, which is the complete divisor. On multiplying this expression by 2 and subtracting, the remainder is zero. Hence the root is $x^2 - 3x + 2$.

In case there are four terms in the root, the sum of the first three, when found as above, is regarded as the new a, called a''. The remaining term is the new b and is called b''. The process is then precisely the same as in the preceding step.

EXERCISES

Find the square roots of the following:

1. $m^2 + 4mn + 6ml + 4n^2 + 12ln + 9l^2.$

2. $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4.$

3. $9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd + 25c^2 + 10cd + d^2.$

4. $9a^2 - 30ab - 3ab^2 + 25b^2 + 5b^3 + \dfrac{b^4}{4}.$

5. $\frac{4}{9}a^2x^4 - \frac{4}{3}abx^3z + \frac{8}{3}a^2bx^2z^2 + b^2x^2z^2 - 4ab^2xz^3 + 4a^2b^2z^4.$

6. $a^2 - 6ab + 10ac - 14ad + 9b^2 - 30bc + 42bd + 25c^2$ $- 70cd + 49d^2.$

7. $\frac{9}{4} + 6x - 17x^2 - 28x^3 + 49x^4.$

8. $9a^6 - 24a^3b^4 - 18a^3c^5 + 6a^3d^2 + 16b^8 + 24b^4c^5 - 8b^4d^2$ $+ 9c^{10} - 6c^5d^2 + d^4.$

9. $25a^{4m}b^{6k} - 70a^{5m}b^{5k} + 49a^{6m}b^{4k}.$

10. $x^{10} - 8x^5w^5 + 16w^{10} - 4x^5y^3 + 16y^3w^5 + 4y^6 + 6x^5z^4$ $- 24z^4w^5 - 12y^3z^4 + 9z^8.$

Find the cube root of each of the following:

11. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2$ $- 3yz^2 + z^3.$

12. $1728x^6 + 1728x^4y^3 + 576x^2y^6 + 64y^9.$

13. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c$ $+ 3bc^2 + c^3.$

14. $8a^3 - 12a^2b + 6ab^2 - b^3.$

15. $8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125.$

16. $27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6.$

17. $1 - 9y^2 + 39y^4 - 99y^6 + 156y^8 - 144y^{10} + 64y^{12}.$

18. $125x^6 - 525x^5y + 60x^4y^2 + 1547x^3y^3 - 108x^2y^4 - 1701xy^5$ $- 729y^6.$

19. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729.$

20. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

21. $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^9$ $- 36a^2b^7 + 9ab^8 - b^9.$

22. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c$ $+ 6bc^2 - c^3.$

23. $343a^6 - 441a^5b + 777a^4b^2 - 531a^3b^3 + 444a^2b^4 - 144ab^5$ $+ 64b^6.$

24. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^5 + 64.$

25. $27l^{12} + 189l^{11} + 198l^{10} - 791l^9 - 594l^8 + 1701l^7 - 729l^6.$

ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

124. The cube root of a number expressed in Arabic figures, as in the case of square root, pp. 225–229, E. C., may be found by the process used for polynomials. An example will illustrate.

Ex. 1. Find the cube root of 389,017.

In order to decide how many digits there are in the root, we observe that $10^3 = 1000$, $100^3 = 1{,}000{,}000$. Hence the root lies between 10 and 100, that is, it contains two digits. Since $70^3 = 343{,}000$ and $80^3 = 512{,}000$, it follows that 7 is the largest number possible in tens' place. The work is arranged as follows:

$$
\begin{array}{rr|ll}
\text{The given cube,} & & 389\,017 & \underline{|70 + 3}, \text{ cube root.} \\
a^3 = 70^3 = & & \underline{343\,000} & \text{1st partial product.} \\
3\,a^2 = 3 \cdot 70^2 = & 14700 & 46\,017 & \text{1st remainder.} \\
3\,ab = 3 \cdot 70 \cdot 3 = & 630 & & \\
b^2 = 3^2 \qquad = & \underline{9} & & \\
3\,a^2 + 3\,ab + b^2 = & 15339 & \underline{46\,017} = b\,(3\,a^3 + 3\,ab + b^2). & \\
 & & 0 &
\end{array}
$$

Having decided as above that the a of the formula is 7 tens, we cube this and subtract, obtaining 46,017 as the remaining part of the power.

The first partial divisor, $3\,a^2 = 14{,}700$, is divided into 46,017, giving a quotient 3, which is the b of the formula. Hence the first complete divisor, $3\,a^2 + 3\,ab + 3\,b^2$, is 15,339 and the product, $b(3\,a^2 + 3\,ab + b^2)$, is 46,017. Since the remainder is zero, the process ends and 73 is the cube root sought.

125. The cube of any number from 1 to 9 contains one, two, or three digits; the cube of any number between 10 and 99 contains four, five, or six digits; the cube of any number between 100 and 999 contains seven, eight, or nine digits, etc. Hence it is evident that if the digits of a number are separated into groups of three figures each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the root.

Ex. 2. Find the cube root of 13,997,521.

$$
\begin{array}{rrl}
 & \text{The given cube, } 13\,997\,521 & \underline{|200 + 40 + 1 = 241, \text{ cube root.}} \\
a^3 = 200^3 = & \underline{8\,000\,000} & \\
3\,a^2 = 120000 & 5\,997\,521 & \\
3\,ab = 24000 & & \\
b^2 = \underline{1600} & & \\
145600 & \underline{5\,824\,000} & = b\,(3\,a^2 + 3\,ab + b^2) \\
3\,a'^2 = 172800 & 173\,521 & \\
3\,a'b' = 720 & & \\
b'^2 = \underline{1} & & \\
173521 & \underline{173\,521} & = b'\,(3\,a'^2 + 3\,a'b' + b'^2). \\
 & 0 &
\end{array}
$$

Since the root contains three digits, the first one is the cube root of 8, the largest integral cube in 13.

The first partial divisor, $3 \cdot 200^2 = 120{,}000$, is completed by adding $3\,ab = 3 \cdot 200 \cdot 40 = 24{,}000$, and $b^2 = 1600$.

The second partial divisor, $3\,a'^2$, which stands for $3(200 + 40)^2 = 172{,}800$, is completed by adding $3\,a'b'$ which stands for $3 \cdot 240 \cdot 1 = 720$, and b'^2 which stands for 1, where a' represents the part of the root *already* found and b' the next digit to be found. At this step the remainder is zero and the root sought is 241.

EXERCISES

Find the square root of each of the following:

1. 58,081.	2. 795,564.	3. 11,641,744.

Find the cube root of each of the following:

4. 110,592.	7. 205,379.	10. 2,146,689.
5. 571,787.	8. 31,855,013.	11. 19,902,511.
6. 7,301,384.	9. 5,929,741.	12. 817,400,375.

126. Since the cube of a decimal fraction has three times as many places as the given decimal, it is evident that the cube root of a decimal fraction contains one decimal place for every three in the cube. Hence for the purpose of determining the places in the root, the decimal part of a cube should be divided into groups of three digits each, counting from the decimal point toward the right.

Ex. Approximate the cube root of 34.567 to two places of decimals.

$$
\begin{array}{rr|ll}
 & & 34.567 & \underline{3 + .2 + .05 + .007} = 3.257 \\
a^3 = 3^3 = & & \underline{27.000} & \\
3\,a^2 = 3 \cdot 3^2 = & 27. & 7.567 & \\
3\,ab = 3 \cdot 3(.2) = & 1.8 & & \\
b^2 = (.2)^2 = & \underline{.04} & & \\
 & 28.84 & \underline{5.768} & = b(3\,a^2 + 3\,ab + b^2) \\
3\,a'^2 = 3(3.2)^2 = & 30.72 & 1.799000 & \\
3\,a'b' = 3(3.2)(.05) = & .48 & & \\
b'^2 = (.05)^2 = & \underline{.0025} & & \\
 & 31.2025 & \underline{1.560125} & = b'(3\,a'^2 + 3\,a'b' + b'^2) \\
3\,a''^2 = 3(3.25)^2 = & 31.6875 & .238875000 & \\
3a''b'' = 3(3.25)(.007) = & .06825 & & \\
b''^2 = (.007)^2 = & \underline{.000049} & & \\
 & 31.755799 & \underline{.222290593} & = b''(3\,a''^2 + 3\,a''b'' + b''^2) \\
 & & .016584407 &
\end{array}
$$

The decimal points are handled exactly as in arithmetic work.

127. Evidently the above process can be carried on indefinitely. 3.257 is an **approximation** to the cube root of 34.567. In fact the cube of 3.257 differs from 34.567 by less than the small fraction .017. The nearest approximation using two decimal places is 3.26. If the third decimal place were any digit less than 5, then 3.25 would be the nearest approximation using two decimal places. Hence three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Approximate the cube root of each of the following to two places of decimals.

1. 21.4736.	6. .003.	11. .004178.
2. 6.5428.	7. .3917.	12. 200.002.
3. 58.	8. .5.	13. 572.274.
4. 2.	9. .05.	14. 31.7246.
5. 3.	10. 6410.37.	15. 54913.416.

16. Approximate the square root in Exs. 1, 2, 10, 11, and 15 of the above list.

CHAPTER VII

QUADRATIC EQUATIONS

EXPOSITION BY MEANS OF GRAPHS

128. We saw, § 65, that a single equation in two variables is satisfied by indefinitely many pairs of numbers. If such an equation is of the **first degree** in the two variables, the graph is in every case a *straight line*.

We are now to consider graphs of equations of the **second degree** in two variables. See § 66.

Ex. 1. Graph the equation $y = x^2$.

By giving various values to x and computing the corresponding values of y, we find pairs of numbers as follows which satisfy this equation:

$$\begin{cases} x = 0, \\ y = 0. \end{cases} \begin{cases} x = 1, \\ y = 1. \end{cases} \begin{cases} x = -1, \\ y = 1. \end{cases} \begin{cases} x = 2, \\ y = 4. \end{cases} \begin{cases} x = -2, \\ y = 4. \end{cases} \begin{cases} x = 3, \\ y = 9. \end{cases} \begin{cases} x = -3, \\ y = 9. \end{cases} \text{etc.}$$

These pairs of numbers correspond to points which lie on a curve as shown in Figure 3.

By referring to the graph the curve is seen to be symmetrical with respect to the y-axis. This can be seen directly from the equation itself since x is involved only as a square and hence, if $y = x^2$ is satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = b$.

It may easily be verified that no three points of this curve lie on a straight line. The curve is called a **parabola**.

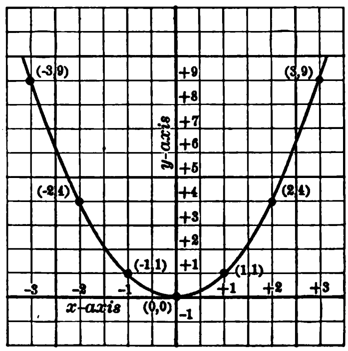


FIG. 3.

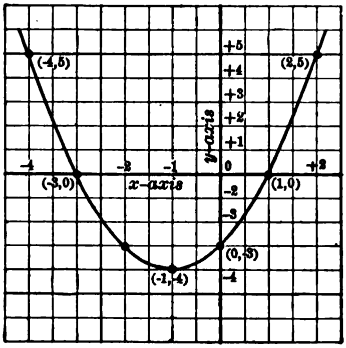


FIG. 4.

Ex. 2. Graph the equation $y = x^2 + 2x - 3$.

Each of the following pairs of numbers satisfies the equation:

$$\begin{cases} x = 0, \\ y = -3. \end{cases} \begin{cases} x = 1, \\ y = 0. \end{cases} \begin{cases} x = -1, \\ y = -4. \end{cases}$$

$$\begin{cases} x = 2, \\ y = 5. \end{cases} \begin{cases} x = -2, \\ y = -3. \end{cases} \begin{cases} x = -3, \\ y = 0. \end{cases}$$

$$\begin{cases} x = -4, \\ y = 5. \end{cases}$$

Plotting these points and drawing a smooth curve through them, we have the graph of the equation, as in Figure 4.

EXERCISES

In this manner graph each of the following:

1. $y = x^2 - 1$.
2. $y = x^2 + 4x$.
3. $y = x^2 + 3x - 4$.
4. $y = x^2 + 5x + 4$.
5. $y = x^2 - 7x + 6$.
6. $y = 3x^2 - 7x + 2$.
7. $y = 5x - x^2 - 4$.
8. $y = 4x - x^2 + 5$.
9. $y = x^2 + 5x - 6$.
10. $y = -x^2 + x$.
11. $y = 4x^2 - 3x - 1$.
12. $y = -4x^2 + 3x + 1$.

129. We now seek to find the points at which each of the above curves cuts the x-axis. The value of y for all points on the x-axis is zero. Hence we put $y = 0$, and try to solve the resulting equation.

Thus in Ex. 2 above, if $y = 0$, $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$, which is satisfied by $x = 1$ and $x = -3$. Hence this curve cuts the x-axis in the two points $x = 1$, $y = 0$ and $x = -3$, $y = 0$, as shown in Figure 4.

Similarly in Ex. 1, if $y = 0$, $x^2 = 0$, and hence $x = 0$. Hence the curve meets the x-axis in the point $x = 0$, $y = 0$, as shown in Figure 3. On this point see § 131, Ex. 2.

EXERCISES

Find the points in which each of the twelve curves in the preceding list cuts the x-axis.

Notice that in every case the expression to the right of the equality sign can be factored, so that when $y = 0$ the resulting equation in x may be solved as in § 94.

Ex. 3. Plot the curve $y = x^2 + 4x + 2$ and find its intersection points with the x-axis.

We are not able to factor $x^2 + 4x + 2$ by inspection. Hence we solve the equation $x^2 + 4x + 2 = 0$ by completing the square as in § 175, E. C., obtaining $x = -2 + \sqrt{2}$ and $x = -2 - \sqrt{2}$. Hence the curve cuts the x-axis in points whose abscissas are given by these values of x.

In making this graph, we first plot points corresponding to *integral* values of x, as before; then, in drawing the smooth curve through these, the intersections made with the x-axis are approximately the points on the number scale corresponding to the *incommensurable* numbers, $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$. See § 109.

EXERCISES

In this manner, find the points at which each of the following curves cuts the x-axis, and plot the curves. For reduction of the results to simplest forms, see §§ 159, 160, E. C.

1. $y = x^2 + 5x + 3$.
2. $y = 3x^2 + 8x - 2$.
3. $y = 6x - 4x^2 + 5$.
4. $y = -4 - 2x + 5x^2$.
5. $y = 2x - 5x^2 + 8$.
6. $y = 5 + 8x - 3x^2$.
7. $y = 3 - 9x^2 - 11x$.
8. $y = -2 - 2x + x^2$.

130. Each of the foregoing exercises involves the solution of an equation of the general form $ax^2 + bx + c = 0$. Obviously, by solving this equation, we shall obtain a formula by means of which every equation of this type may be solved. See § 179, E. C.

The two values of x are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

EXERCISES

By means of this formula, find the solutions of each of the following equations:

1. $2x^2-3x-4=0.$
2. $3x^2+2x-1=0.$
3. $3x^2-2x-1=0.$
4. $4x^2+6x+1=0.$
5. $x^2-7x+12=0.$
6. $5x^2+8x+3=0.$
7. $5x^2-8x+3=0.$
8. $5x^2+8x-3=0.$
9. $5x^2-8x-3=0.$
10. $2x-3x^2+7=0.$
11. $3x-9x^2+1=0.$
12. $7x^2-3x-2=0.$
13. $6x^2+7x+1=0.$
14. $4x^2+5x-3=0.$
15. $4x^2-5x-3=0.$
16. $8x^2+3x-5=0.$
17. $7x^2+x-3=0.$
18. $7x^2-x-4=0.$
19. $x^2-2ax=3b-a^2.$
20. $x^2-6ax=49c^2-9a^2.$
21. $x^2+\dfrac{a(a+b)}{3}=ax+\dfrac{(a+b)x}{3}.$
22. $-2x^2-\dfrac{c-d}{2}x-2c^2x=\dfrac{c^2(c-d)}{2}.$
23. $x^2-\dfrac{mx}{2}+2mn=4nx.$
24. $x^2-2ax+4ab=b^2+3a^2.$
25. $x^2-abx+a^2b-ax=ab^2-bx.$
26. $x^2+9-c=6x.$
27. $nx^2+m^2n=mn^2x+mx.$
28. $2(a+1)x^2-(a+1)^2x+2(a+1)=4x.$
29. $x^2+9cd+3c=(3c+3d+1)x.$
30. $x^2+2a^2+3a-2=(3a+1)x$

131. We now consider the intersections of other straight lines besides the x-axis with curves like those plotted above.

Ex. 1. Graph on the same axes the straight line, $y=-2$ and the curve, $y=x^2+2x-3$.

This line is parallel to the x-axis and two units below it. It cuts the curve in the two points whose abscissas are $x_1=-1+\sqrt{2}$ and $x_2=-1-\sqrt{2}$, as found by substituting -2 for y in $y=x^2+2x-3$ and solving the resulting equation in x by the formula, § 130.

Ex. 2. Graph on the same axes $y=-4$ and $y=x^2+2x-3$.

This line *seems not to cut* the curve but to *touch* it at the point whose abscissa is $x=-1$.

Substituting and solving as before, we find,

$$x_1=\frac{-2+\sqrt{4-4}}{2}=\frac{-2+0}{2}=-1$$

and
$$x_2=\frac{-2-\sqrt{4-4}}{2}=\frac{-2-0}{2}=-1.$$

In this case the two values of x are *equal*, and there is only *one* point common to the line and the curve. This is understood by thinking of the line $y=-2$, in the preceding example, as moved down to the position $y=-4$, whereupon the two values of x which were *distinct* now *coincide*.

132. From the formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, it is clear that the general equation, $ax^2+bx+c=0$ has two *distinct* solutions unless the expression b^2-4ac reduces to zero, in which case the two values of x *coincide*, giving $x_1=\dfrac{-b+0}{2a}=-\dfrac{b}{2a}$ and $x_2=\dfrac{-b-0}{2a}=-\dfrac{b}{2a}$.

Ex. 1. In $2x^2-9x+8=0$, determine without solving whether the two values of x are distinct or coincident.

In this case, $a=2,\ b=-9,\ c=8.$

Hence $b^2-4ac=81-64=17.$

Hence the values of x are distinct.

Ex. 2. In $4x^2-12x+9=0$, determine whether the values of x are distinct or coincident.

In this case, $b^2-4ac=144-4\cdot4\cdot9=0$. Hence the values of x are *coincident*.

EXERCISES

In each of the following, determine without solving whether the two solutions are distinct or coincident:

1. $x^2 - 7x + 4 = 0.$
2. $4x^2 + 28x + 49 = 0.$
3. $9x^2 + 12x + 4 = 0.$
4. $x^2 + 6x + 9 = 0.$
5. $-x^2 + 9x + 25 = 0.$
6. $6x^2 - 3x - 1 = 0.$
7. $4x^2 - 16x + 16 = 0.$
8. $8x^2 - 13 = 4x.$
9. $12x^2 - 18 = 24x.$
10. $16x^2 - 56x = -49.$

133. Definition. A line which cuts a curve in two *coincident points* is said to be **tangent to the curve.**

134. Problem. What is the value of a in $y = a$, if this line is tangent to the curve $y = x^2 + 5x + 8$?

Substituting a for y and solving by means of the formula, we have

$$x = \frac{-5 \pm \sqrt{25 - 4(8-a)}}{2}.$$

If the line is to be tangent to the curve, then the expression under the radical sign must be zero so that the two values of x may coincide. That is, $25 - 4(8 - a) = 0$, or $a = \frac{7}{4}$.

On plotting the curve, the line $y = \frac{7}{4}$ is found to be tangent to it.

EXERCISES

In the first 18 exercises on p. 386 obtain equations of curves by letting the left members equal y. Then find the equations of straight lines, $y = a$, which are tangent to these curves.

135. Problem. Find the intersection points of the curve $y = x^2 + 3x + 5$ and the line $y = 2\frac{1}{2}$.

Substituting for y and solving for x we have

$$x_1 = \frac{-6 + \sqrt{36 - 40}}{4} = \frac{-6 + 2\sqrt{-1}}{4} = \frac{-3 + \sqrt{-1}}{2};$$

$$x_2 = \frac{-6 - \sqrt{36 - 40}}{4} = \frac{-6 - 2\sqrt{-1}}{4} = \frac{-3 - \sqrt{-1}}{2}.$$

These results involve the **imaginary unit** already noticed in § 112. Numbers of the type $a + b\sqrt{-1}$ are discussed further in § 195. For the present we will regard such results as merely indicating that the conditions stated by the equations cannot be fulfilled by *real numbers*. This means that the curve and the line have *no point in common*, as is evident on constructing the graphs.

By proceeding as in § 134 we find that the line $y = \frac{11}{4}$ is *tangent* to the curve $y = x^2 + 3x + 5$. Clearly all lines $y = a$, in which $a > \frac{11}{4}$, are *above* this line and hence cut this curve in *two points*.

All such lines for which $a < \frac{11}{4}$ are *below* the line $y = \frac{11}{4}$ and hence do *not meet* the curve at all.

Solving $y = a$ and $y = x^2 + 3x + 5$ for x by first substituting a for y we have

$$x = \frac{-3 \pm \sqrt{4a - 11}}{2}.$$

If $a > \frac{11}{4}$ the number under the radical sign is *positive*, and there are *two real and distinct* values of x. Hence the line and the curve meet in two points.

If $a < \frac{11}{4}$, the number under the radical sign is *negative*. Consequently the values of x are *imaginary* and the line and the curve do not meet.

Hence we see that the conclusions obtained from the solution of the equations agree with those obtained from the graphs.

136. From the two preceding problems it appears that it is possible to determine the *relative* positions of the line and the curve *without completely solving* the equations. Namely, as soon as y is eliminated and the equation in x is reduced to the form $ax^2 + bx + c = 0$, we examine $b^2 - 4ac$ as follows:

(1) If $b^2 - 4ac > 0$, *i.e. positive,* then the line cuts the curve in two distinct points.

(2) If $b^2 - 4ac = 0$, then the line is tangent to the curve. See § 133.

(3) If $b^2 - 4ac < 0$, *i.e. negative,* then the line does not cut the curve.

137. Problem. Find the points of intersection of

$$y = x^2 + 3x + 13 \ (1), \text{ and } y + 3x = 7 \ (2).$$

Eliminating y and reducing the resulting equation in x to the form $ax^2 + bx + c = 0$, we have $x^2 + 6x + 6 = 0$.

Solving, $x_1 = -3 + \sqrt{3}$, $x_2 = -3 - \sqrt{3}$.

Substituting these values of x in (2) and solving for y, we have

$$\begin{cases} x_1 = -3 + \sqrt{3} \\ y_1 = 16 - 3\sqrt{3} \end{cases} \text{ and } \begin{cases} x_2 = -3 - \sqrt{3} \\ y_2 = 16 + 3\sqrt{3} \end{cases}$$

which are the points in which the line meets the curve.

Here $b^2 - 4ac = 12$, which shows in advance that there are *two points* of intersection.

EXERCISES

In each of the following determine without graphing whether or not the line meets the curve, and in case it does, find the intersection points:

1. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

2. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2y = x - 1. \end{cases}$

3. $\begin{cases} y = 3x^2 - 2x - 1, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y = 4x^2 + 6x + 1, \\ x = y + 5. \end{cases}$

5. $\begin{cases} y = x^2 - 7x + 12, \\ 5x - y = -1. \end{cases}$

6. $\begin{cases} y = 5x^2 + 8x + 3, \\ 2y - 5x - 2 = 0. \end{cases}$

7. $\begin{cases} y = 5x^2 - 8x + 3, \\ 3 - x = 3y. \end{cases}$

8. $\begin{cases} y = -5x^2 + 8x - 3, \\ 2 - 4y - x = 0. \end{cases}$

9. $\begin{cases} y = -5x^2 - 8x - 3, \\ 5y - 3x = 8. \end{cases}$

10. $\begin{cases} y = 3x - 3x^2 + 7, \\ -5 - 3x + 2y = 0. \end{cases}$

138. Problem. Graph the equation $x^2 + y^2 = 25$.

Writing the equation in the form $y = \pm\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$\begin{cases} x = 0, \\ y = \pm 5. \end{cases}$ $\begin{cases} x = \pm 5, \\ y = 0. \end{cases}$ $\begin{cases} x = 3, \\ y = \pm 4. \end{cases}$ $\begin{cases} x = -3, \\ y = \pm 4. \end{cases}$ $\begin{cases} x = 4, \\ y = \pm 3. \end{cases}$ $\begin{cases} x = -4, \\ y = \pm 3. \end{cases}$

Evidently, for x greater than 5 in absolute value, the corresponding y's are *imaginary*, and for each x between -5 and $+5$ there are two y's equal in absolute value, but with opposite signs.

It seems apparent that these points lie on the circumference of a circle whose radius is 5, as shown in Figure 5. Indeed, if we consider any point x_1, y_1 on this circumference, it is evident that $x_1^2 + y_1^2 = 25$, since the sum of the squares on the sides of a right triangle is equal to the square on the hypotenuse. (See figure, p. 207, E. C.)

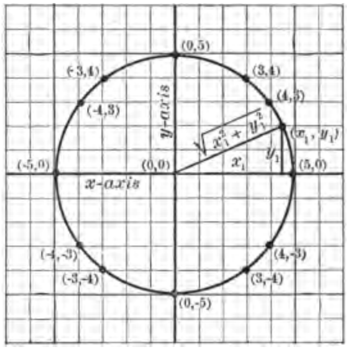


FIG. 5.

The equation $x^2 + y^2 = 25$ is, therefore, the equation of a circle with radius 5. Similarly, $x^2 + y^2 = r^2$ is the equation of a circle with center at the point (0, 0) and radius r.

139. Problem. Find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

Eliminating y from these equations, and reducing the equation in x to the form $ax^2 + bx + c = 0$, we have

$$x^2 - 7x + 12 = 0.$$

From which $$x_1 = 4, \ x_2 = 3.$$

Substituting these values of x in $x + y = 7$, we have $y_1 = 3$, $y_2 = 4$. Hence $x_1 = 4$, $y_1 = 3$ and $x_2 = 3$, $y_2 = 4$ are the required points.

Verify this by graphing the two equations on the same axes.

140. Problem. Find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 25$.

Eliminating y and solving for x, we find $x = \dfrac{6 \pm 0}{2} = 3.$

Hence $x_1 = x_2 = 3$, from which $y_1 = y_2 = 4$.

Since the two values of x coincide, and likewise the two values of y, the circumference and the line have but *one point* in common. Verify by graphing the line and the circle on the same axes.

141. Problem. Find the points of intersection of

$$x^2+y^2=25$$
$$\text{and } x+y=10.$$

Substituting for y and solving for x we have

$$x=\frac{20\pm\sqrt{400-600}}{4}=\frac{20\pm\sqrt{-200}}{4}$$
$$=\frac{20\pm10\sqrt{-2}}{4}=\frac{10\pm5\sqrt{-2}}{2}.$$

The imaginary values of x indicate that there is no intersection point. Verify by plotting.

EXERCISES

In each of the following determine by solving whether the line and the circumference meet, and in case they do, find the points of intersection. Verify each by constructing the graph.

1. $\begin{cases} x^2+y^2=16, \\ x+y=4. \end{cases}$

2. $\begin{cases} x^2+y^2=36, \\ 4x+y=6. \end{cases}$

3. $\begin{cases} x^2+y^2=25, \\ 2x+y=-5. \end{cases}$

4. $\begin{cases} x^2+y^2=20, \\ 2x+y=0. \end{cases}$

5. $\begin{cases} x^2+y^2=7, \\ x+y=8. \end{cases}$

6. $\begin{cases} x^2+y^2=8, \\ x-y=4. \end{cases}$

7. $\begin{cases} x^2+y^2=41, \\ x-3y=7. \end{cases}$

8. $\begin{cases} x^2+y^2=29, \\ 3x-7y=-29. \end{cases}$

9. $\begin{cases} x^2+y^2=12, \\ x-y=6. \end{cases}$

10. $\begin{cases} x^2+y^2=4, \\ 2x-3y=4. \end{cases}$

11. $\begin{cases} x^2+y^2=40, \\ x+2y=10. \end{cases}$

12. $\begin{cases} x^2+y^2=25, \\ x+y=9. \end{cases}$

142. Problem. Graph on the same axes the circle, $x^2+y^2=5^2$, and the lines, $3x+4y=20$, $3x+4y=25$, and $3x+4y=30$.

The first line *cuts* the circumference in two distinct points, the second seems to be *tangent* to it, and the third does *not meet* it. Observe that the three lines are parallel. See Figure 6.

In order to discuss the relative positions of such straight lines and the circumference of a circle, we solve the following equations simultaneously:

$$x^2 + y^2 = r^2 \qquad (1)$$

$$3x + 4y = c \qquad (2)$$

Eliminating y by substitution, and solving for x, we find

$$x = \frac{3c \pm 4\sqrt{25r^2 - c^2}}{25}. \qquad (3)$$

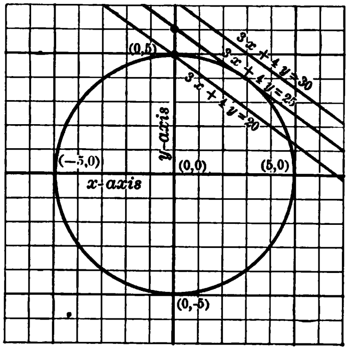


FIG. 6.

The two values of x from (3) are the abscissas of the points of intersection of the circumference (1) and the line (2).

These values of x are *real and distinct* if $25r^2 - c^2$ is *positive*, *real and coincident* if $25r^2 - c^2 = 0$, and *imaginary* if $25r^2 - c^2$ is *negative*.

Now $25r^2 - c^2$ is *positive* if $r = 5$, $c = 20$; *zero* if $r = 5$, $c = 25$; and *negative* if $r = 5$, $c = 30$.

Hence these results obtained from the solution of the equations agree with the facts observed in the graphs above.

143. Definition. Letters such as c and r in the above solution to which any arbitrary constant values may be assigned are called **parameters**, while x and y are the **unknowns** of the equations.

EXERCISES

Solve each of the following pairs of equations.

Give such values to the parameters involved that the line (*a*) may cut the curve in two distinct points, (*b*) may be tangent to the curve, (*c*) shall fail to meet the curve.

1. $\begin{cases} x^2 + y^2 = 4, \\ ax + 3y = 16. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 16, \\ 2x + by = 12. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 2x + 3y = c. \end{cases}$

4. $\begin{cases} y^2 = 8x, \\ 3x + 4y = c. \end{cases}$

5. $\begin{cases} 5y^2 = 2px, \\ x + y = 1. \end{cases}$

6. $\begin{cases} y = x^2 + mx + 4, \\ x + y = 4. \end{cases}$

7. $\begin{cases} y = mx^2 - x - 4, \\ x - 3y = 8. \end{cases}$

8. $\begin{cases} y = 2x^2 - 3x + 1, \\ 2x - by - 1 = 0. \end{cases}$

9. $\begin{cases} y = 3x^2 + mx, \\ x + y + 3 = 0. \end{cases}$

10. $\begin{cases} y = mx^2 + 2x, \\ 2y - bx - 5 = 0. \end{cases}$

11. $\begin{cases} y = x^2 + nx + 1, \\ ax + 2y = 10. \end{cases}$

12. $\begin{cases} x^2 + y^2 = r^2, \\ ax + by = c. \end{cases}$

144. Problem. Graph the equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1.$

Writing the equation in the form $y = \pm \frac{4}{5}\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$\begin{cases} x = 0, \\ y = \pm 4, \end{cases}$ $\begin{cases} x = \pm 5, \\ y = 0, \end{cases}$ $\begin{cases} x = 1, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = -1, \\ y = \pm 3.9. \end{cases}$

$\begin{cases} x = 2, \\ y = \pm 3.7, \end{cases}$ $\begin{cases} x = 3, \\ y = \pm 3.2, \end{cases}$ $\begin{cases} x = 4, \\ y = \pm 2.4. \end{cases}$

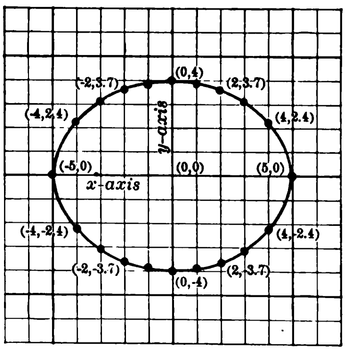


FIG. 7.

Evidently if x is greater than 5 in absolute value, the corresponding values of y are imaginary.

Plotting these points, they are found to lie on the curve shown in Figure 7. This curve is called an **ellipse**.

EXERCISES

Solve the following pairs of equations.

In this way determine whether the straight line and the curve intersect, and in case they do, determine the coördinates of the intersection points. Verify each by constructing the graphs.

1. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1. \\ 3x + 4y = 12. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{49} + \dfrac{y^2}{16} = 1, \\ 2x - 7y = 8. \end{cases}$

3. $\begin{cases} x^2+4y^2=25, \\ 2x-y=4. \end{cases}$

4. $\begin{cases} 3x^2+2y^2=11, \\ x-3y=7. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{9}=1, \\ 2x-y=14. \end{cases}$

6. $\begin{cases} y=2x^2-3x+4, \\ y-4x-8=0. \end{cases}$

7. $\begin{cases} x^2+y^2=16, \\ x+y=7. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{64}+\dfrac{y^2}{12}=1, \\ 4y-2x=4. \end{cases}$

9. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{45}=1, \\ -5x+6y=10. \end{cases}$

10. $\begin{cases} \dfrac{x^2}{49}+\dfrac{y^2}{25}=1, \\ x+y=12. \end{cases}$

When arbitrary constants are introduced in the equations of a straight line and an ellipse, we may determine values for these constants so as to make the line cut the ellipse, touch it, or not cut it, as in the case of the circle, § 142.

EXERCISES

Solve each of the following pairs simultaneously.

Give such values to the constants that the line shall (*a*) cut the curve in two distinct points, (*b*) be a tangent to the curve, (*c*) have no point in common with the curve.

In case (*b*) is found very difficult, this may be omitted.

1. $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{16}=1, \\ 8x+5y=40. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{b^2}=1, \\ 4x+15y=60. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{16}=1, \\ 4x-5y=c. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{25}=1, \\ 5x-by=20. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{25}=1, \\ ax+4y=20. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ ax+6y-60=0. \end{cases}$

7. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{25}=1, \\ 5x+by=60. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \\ bx-2y=5. \end{cases}$

9. $\begin{cases} x^2+y^2=r^2, \\ ax-3y=4. \end{cases}$

10. $\begin{cases} 5x^2+3y^2=16, \\ hx-ky=8. \end{cases}$

11. $\begin{cases} x^2+7y^2=144, \\ ax+by=12. \end{cases}$

12. $\begin{cases} x^2+4y^2=r^2, \\ ax+by=c. \end{cases}$

SPECIAL METHODS OF SOLUTION

145. We have thus far solved simultaneously one equation of the second degree with one of the first degree. After substitution each has reduced to the solution of an ordinary quadratic, namely, of the form, $ax^2 + bx + c = 0$.

While this is an effective general method, yet some important special forms of solution are shown in the following examples:

Ex. 1. Solve $\begin{cases} x^2 + y^2 = a, & (1) \\ x - y = b. & (2) \end{cases}$

Square both members of (2) and subtract from (1).

$$2xy = a - b^2. \qquad (3)$$

Add (1) and (3). $x^2 + 2xy + y^2 = 2a - b^2.$ (4)

Hence $$x + y = \pm\sqrt{2a - b^2}. \qquad (5)$$

From (2) and (5), adding and subtracting

$$\begin{cases} x_1 = \dfrac{\sqrt{2a - b^2} + b}{2}, \\ \\ y_1 = \dfrac{\sqrt{2a - b^2} - b}{2}, \end{cases} \text{ and } \begin{cases} x_2 = \dfrac{-\sqrt{2a - b^2} + b}{2}, \\ \\ y^2 = \dfrac{-\sqrt{2a - b^2} - b}{2}. \end{cases}$$

Ex. 2. Solve $\begin{cases} x^2 - y^2 = a, & (1) \\ x - y = b. & (2) \end{cases}$

From (1) $$(x - y)(x + y) = a. \qquad (3)$$

Substituting b for $x - y$ in (3), $x + y = \dfrac{a}{b}.$ (4)

Then (2) and (4) may be solved as above.

Ex. 3. Solve $\begin{cases} x + y = a, & (1) \\ xy = b. & (2) \end{cases}$

Multiply (2) by 4, subtract from the square of (1), and get

$$x^2 - 2xy + y^2 = a^2 - 4b, \qquad (3)$$

whence, $$x - y = \pm\sqrt{a^2 - 4b}. \qquad (4)$$

Then (1) and (4) may be solved as in Ex. 1.

The equations $\begin{cases} x - y = a, \\ xy = b, \end{cases}$

may be solved in a similar manner.

146. We are now to study the solution of a pair of equations each of the second degree. See § 66.

Consider
$$x^2 + y = a, \qquad (1)$$
$$x + y^2 = b. \qquad (2)$$

Solving (1) for y and substituting in (2) we have,
$$x + a^2 - 2\,ax^2 + x^4 = b,$$
which is of the fourth degree and cannot be solved by any methods thus far studied. There are, however, special cases in which two equations each of the second degree can be solved by a proper combination of methods already known.

147. Case I. *When only the squares of the unknowns enter the equations.*

Example. Solve $\begin{cases} a_1x^2 + b_1y^2 = c_1, \\ a_2x^2 + b_2y^2 = c_2. \end{cases}$

These equations are *linear* if x^2 and y^2 are regarded as the unknowns.

Solving for x^2 and y^2 as in § 73, we obtain,
$$x^2 = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},\ y^2 = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Hence, taking square roots,
$$\begin{cases} x_1 = \sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_1 = \sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases} \qquad \begin{cases} x_3 = \sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_3 = -\sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases}$$
$$\begin{cases} x_2 = -\sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_2 = \sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases} \qquad \begin{cases} x_4 = -\sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_4 = -\sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}. \end{cases}$$

In this case there are four pairs of numbers which satisfy the two equations. This is in general true of two equations each of the second degree.

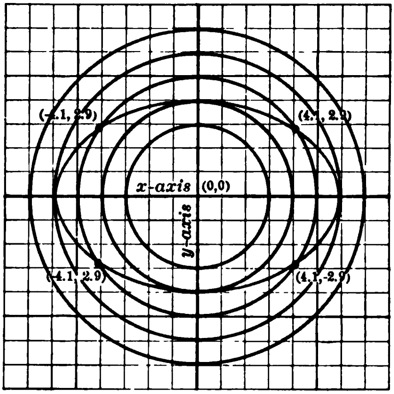


FIG. 8.

Example. Solve simultaneously, obtaining results to one decimal place:

$$\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, & (1) \\ x^2+y^2=25. & (2) \end{cases}$$

Clear (1) of fractions and proceed as above. Verify the solution by reference to the graph given in Figure 8.

EXERCISES

Solve simultaneously each of the following pairs of equations and interpret all the solutions in each case from the graph in Figure 8:

1. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ x^2+y^2=36. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ x^2+y^2=16. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ x^2+y^2=49. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ x^2+y^2=9. \end{cases}$

148. Problem. Graph the equation $\dfrac{x^2}{25}-\dfrac{y^2}{16}=1.$

Writing the equation in the form $y=\pm\frac{4}{5}\sqrt{x^2-25}$, and assigning values to x, we compute the corresponding values of y exactly or approximately as follows:

$\begin{cases} x=\pm 5, \\ y=0, \end{cases}$ $\begin{cases} x=6\frac{1}{4}, \\ y=\pm 3, \end{cases}$ $\begin{cases} x=-6\frac{1}{4}, \\ y=\pm 3, \end{cases}$ $\begin{cases} x=7, \\ y=\pm 3.9, \end{cases}$ $\begin{cases} x=-7, \\ y=\pm 3.9, \end{cases}$ $\begin{cases} x=8, \\ y=\pm 5, \end{cases}$ $\begin{cases} x=-8, \\ y=\pm 5. \end{cases}$

Evidently when x is less than 5 in absolute value, y is imaginary, and as x increases beyond 8 in absolute value, y continually increases.

Plotting these points, they are found to lie on the curve as shown in Figure 9. This curve is called a **hyperbola**.

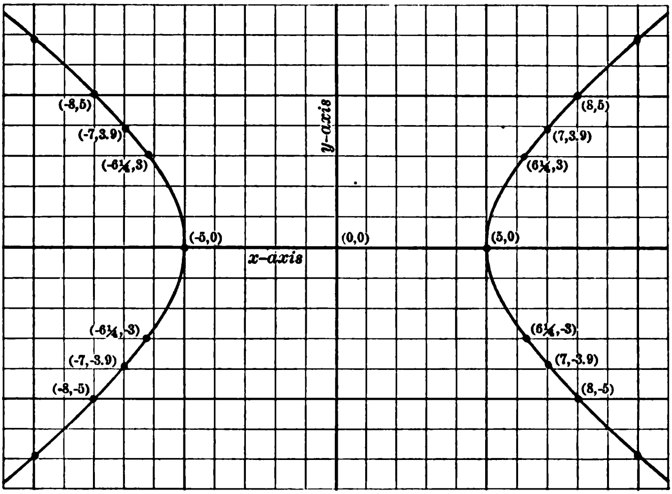


FIG. 9.

EXERCISES

Solve each of the following pairs of equations.

Construct a graph similar to the one in Figure 8 which shall contain the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ and the circles given in Exs. 1, 2, and 3. Construct another graph containing the same hyperbola and the ellipses given in Exs. 4, 5, and 6. From these graphs interpret the solutions of each pair of equations.

1. $\begin{cases}\frac{x^2}{25}-\frac{y^2}{16}=1,\\ x^2+y^2=16.\end{cases}$ 2. $\begin{cases}\frac{x^2}{25}-\frac{y^2}{16}=1,\\ x^2+y^2=25.\end{cases}$ 3. $\begin{cases}\frac{x^2}{25}-\frac{y^2}{16}=1,\\ x^2+y^2=36.\end{cases}$

4. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1. \end{cases}$ 5. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1. \end{cases}$ 6. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1. \end{cases}$

7. Graph the equation $xy = 9$.

Graph $xy = 8$ on the same axes with each of the following:

8. $x^2 + y^2 = 16$. 9. $x^2 + y^2 = 25$. 10. $x^2 + y^2 = 4$.

11. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. 12. $\dfrac{x^2}{25} + \dfrac{y^2}{10.24} = 1$. 13. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

14. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$. 15. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$. 16. $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$.

17. From those graphs in Exs. 8 to 16, in which the curves meet, determine as accurately as possible by measurement the coördinates of the points of intersection or tangency.

18. Solve simultaneously the pairs of equations given in Exs. 8 to 10, after studying the method explained in § 150, Ex. 1. Compare the results with those obtained from the graphs.

19. Solve Exs. 11 to 16 by the method explained in § 149, and compare the results with those obtained from the graphs.

149. Case II. *When all terms containing the unknowns are of the second degree in the unknowns.*

Example. Solve
$$\begin{cases} 2x^2 - 3xy + 4y^2 = 3, & (1) \\ 3x^2 - 4xy + 3y^2 = 2. & (2) \end{cases}$$

Put $y = vx$ in (1) and (2), obtaining

$$\begin{cases} x^2(2 - 3v + 4v^2) = 3, & (3) \\ x^2(3 - 4v + 3v^2) = 2. & (4) \end{cases}$$

Hence from (3) and (4),

$$x^2 = \frac{3}{2 - 3v + 4v^2}, \text{ and also } x^2 = \frac{2}{3 - 4v + 3v^2}. \qquad (5)$$

From (5) $$\frac{3}{2-3v+4v^2}=\frac{2}{3-4v+3v^2}, \tag{6}$$

or $$v^2-6v+5=0. \tag{7}$$

Hence $$v=1, \text{ and } v=5. \tag{8}$$

From $y=vx$, $$y=x, \text{ and } y=5x. \tag{9}$$

If $y=x$, then from (1) and (2),

$$\begin{cases} x=1, \\ y=1, \end{cases} \text{and} \begin{cases} x=-1, \\ y=-1. \end{cases}$$

If $y=5x$, then from (1) and (2),

$$\begin{cases} x=\dfrac{1}{\sqrt{29}}, \\ y=\dfrac{5}{\sqrt{29}}, \end{cases} \text{and} \begin{cases} x=-\dfrac{1}{\sqrt{29}}, \\ y=-\dfrac{5}{\sqrt{29}}. \end{cases}$$

Verify each of these four solutions by substituting in equations (1) and (2).

150. There are many other special forms of simultaneous equations which can be solved by proper combination of the methods thus far used. Also, many pairs of equations of a degree higher than the second in the two unknowns may be solved by means of quadratic equations.

The suggestions given in the following examples illustrate the devices in most common use.

The solution should in each case be completed by the student.

Ex. 1. Solve
$$\begin{cases} x^2+y^2=58, & (1) \\ xy=21. & (2) \end{cases}$$

Adding twice (2) to (1) and taking square roots, we have

$$x+y=10, \text{ and } x+y=-10. \tag{3}$$

Each of the equations (3) may now be solved simultaneously with (2), as in Ex. 3, p. 396.

Ex. 2. Solve
$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=5, & (1) \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=13. & (2) \end{cases}$$

Let $\dfrac{1}{x}=a$ and $\dfrac{1}{y}=b$. Then these equations reduce to
$$\begin{cases} a+b=5, & (3) \\ a^2+b^2=13. & (4) \end{cases}$$

(3) and (4) may then be solved as in Ex. 1, p. 396.

Ex. 3. Solve
$$\begin{cases} x^2+y^2+x+y=8, & (1) \\ xy=2. & (2) \end{cases}$$

Add twice (2) to (1), obtaining
$$x^2+2\,xy+y^2+x+y=12. \qquad (3)$$

Let $x+y=a$. Then (3) reduces to
$$a^2+a=12,$$
or,
$$a=3,\ a=-4. \qquad (4)$$
Hence
$$x+y=3,\ \text{and}\ x+y=-4. \qquad (5)$$
Now solve each equation in (5) simultaneously with (2).

Ex. 4. Solve
$$\begin{cases} x^4y^4+x^2y^2=272, & (1) \\ x^2+y^2=10. & (2) \end{cases}$$

In (1) substitute a for x^2y^2. Then
$$a^2+a=272,\ \text{whence}\ a=16,\ \text{and}\ -17.$$

Hence
$$xy=\pm\sqrt{16}=\pm 4,\ \text{and}\ \pm\sqrt{-17}.$$

Each of these equations may now be solved simultaneously with (2), as in Ex. 1, p. 401.

Ex. 5. Solve
$$\begin{cases} x^3-y^3=117, & (1) \\ x-y=3. & (2) \end{cases}$$

By factoring, (1) becomes

$$(x-y)(x^2+xy+y^2)=117. \tag{3}$$

Substituting 3 for $x-y$, we have

$$x^2+xy+y^2=39. \tag{4}$$

(2) and (4) may now be solved by substitution as in §§ 140–144.

Ex. 6. Solve
$$\begin{cases} x^3+y^3=513, & (1) \\ x+y=9. & (2) \end{cases}$$

Factor (1) and substitute 9 for $x+y$. Then proceed as in Ex. 5.

Ex. 7. Solve
$$\begin{cases} x^2y+xy^2=126, & (1) \\ x+y=9. & (2) \end{cases}$$

Factoring (1) and substituting 9 for $x+y$, we have

$$xy=14. \tag{3}$$

(2) and (3) may then be solved as in Ex. 3, p. 396.

Ex. 8. Solve
$$\begin{cases} x^3+y^3=54\,xy, & (1) \\ x+y=6. & (2) \end{cases}$$

Factor (1) and substitute 6 for $x+y$, obtaining

$$x^2-xy+y^2=9\,xy. \tag{3}$$

(2) and (3) may now be solved by substitution, as in §§ 140–144.

Ex. 9. Solve
$$\begin{cases} x^3-y^3=63, & (1) \\ x^2+xy+y^2=21. & (2) \end{cases}$$

Factor (1) and substitute 21 for x^2+xy+y^2, then proceed as in Ex. 8.

Ex. 10. Solve
$$\begin{cases} x^3+y^3=243, & (1) \\ x^2y+xy^2=162. & (2) \end{cases}$$

Multiply (2) by 3 and add to (1), obtaining a perfect cube. Taking cube roots, we have

$$x+y=9. \tag{3}$$

(1) and (3) are now solved as in the preceding example.

Ex. 11. Solve
$$\begin{cases} x^4 + y^4 = 641, & (1) \\ x + y = 7. & (2) \end{cases}$$

Raise (2) to the fourth power and subtract (1), obtaining

$$4x^3y + 6x^2y^2 + 4xy^3 = 1760. \qquad (3)$$

Factoring, $\quad 2xy(2x^2 + 3xy + 2y^2) = 1760. \qquad (4)$

Squaring (2) we have

$$2x^2 + 4xy + 2y^2 = 98, \qquad (5)$$

or
$$2x^2 + 3xy + 2y^2 = 98 - xy. \qquad (6)$$

Substituting (6) in (4), we have

$$2xy(98 - xy) = 1760, \qquad (7)$$

or
$$x^2y^2 - 98xy + 880 = 0. \qquad (8)$$

In (8) put $xy = a$, obtaining

$$a^2 - 98a + 880 = 0. \qquad (9)$$

The solution of (9) gives two values for xy, each of which may now be combined with (2) as in Ex. 3, p. 396.

EXERCISES

Solve each of the following pairs of equations:

1. $\begin{cases} r^2 + rs + s^2 = 63, \\ r - s = 3. \end{cases}$

2. $\begin{cases} 3x^2 + 2y^2 = 35, \\ 2x^2 - 3y^2 = 6. \end{cases}$

3. $\begin{cases} 3x^2 + 2xy = 16, \\ 4x^2 - 3xy = 10. \end{cases}$

4. $\begin{cases} a^2 + ab + b^2 = 7, \\ a^2 - ab + b^2 = 19. \end{cases}$

5. $\begin{cases} x^2 + y^2 = a, \\ xy = b. \end{cases}$

6. $\begin{cases} x^2 + y^2 = a, \\ x^2 + z^2 = b, \\ y^2 + z^2 = c. \end{cases}$

7. $\begin{cases} ax - by = 0, \\ x^2 + y^2 = c. \end{cases}$

8. $\begin{cases} x^2 + xy = a, \\ y^2 + xy = b. \end{cases}$

9. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 7. \end{cases}$

10. $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$

11. $\begin{cases} x^2 - 3xy = 0, \\ 5x^2 + 3y^2 = 9. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 19, \\ \dfrac{1}{x} + \dfrac{1}{y} = 1. \end{cases}$

13. $\begin{cases} 3x - 2y = 6, \\ 3x^2 - 2xy + 4y^2 = 12. \end{cases}$

14. $\begin{cases} a + b + ab = 11, \\ (a + b)^2 + a^2b^2 = 61. \end{cases}$

15. $\begin{cases} \dfrac{1}{a^2} + \dfrac{1}{ab} + \dfrac{1}{b^2} = 49, \\ \dfrac{1}{a} + \dfrac{1}{b} = 8. \end{cases}$

16. $\begin{cases} 4a^2 - 2ab = b^2 - 16, \\ 5a^2 = 7ab - 36. \end{cases}$

17. $\begin{cases} 3x^2 - 9y^2 = 12, \\ 2x - 3y = 14. \end{cases}$

18. $\begin{cases} x^2 + xy + y^2 = a, \\ x^2 + y^2 = b. \end{cases}$

19. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

20. $\begin{cases} x^2 + y^2 + x - y = 36, \\ xy = 15. \end{cases}$

21. $\begin{cases} x^2 - 5xy + y^2 = -2, \\ x^2 + 7xy + y^2 = 22. \end{cases}$

22. $\begin{cases} a^2 + 6ab + b^2 = 124, \\ a + b = 8. \end{cases}$

23. $\begin{cases} a^2 - 3ab + 2b^2 = 0, \\ 2a^2 + ab - b^2 = 9. \end{cases}$

24. $\begin{cases} x^2 + y^2 + 2x + 2y = 27, \\ xy = -12. \end{cases}$

25. $\begin{cases} x^2 + y^2 - 5x - 5y = -4, \\ xy = 5. \end{cases}$

26. $\begin{cases} (7 + x)(6 + y) = 80, \\ x + y = 5. \end{cases}$

27. $\begin{cases} (x - 4)^2 + (y + 4)^2 = 100, \\ x + y = 14. \end{cases}$

28. $\begin{cases} xy + y + x = 17, \\ x^2y^2 + y^2 + x^2 = 129. \end{cases}$

29. $\begin{cases} b + a^2 = 5(a - b), \\ a + b^2 = 2(a - b). \end{cases}$

30. $\begin{cases} (13x)^2 + 2y^2 = 177, \\ (2y)^2 - 13x^2 = 3. \end{cases}$

31. $\begin{cases} \left(\dfrac{9}{x}\right)^2 = \left(\dfrac{25}{y}\right)^2 - 16, \\ \dfrac{9}{x^2} = \dfrac{25}{y^2}. \end{cases}$

32. $\begin{cases} x^2 + y^2 = 20, \\ 5x^2 - 3y^2 = 28. \end{cases}$

33. $\begin{cases} x^2 = -5 - 3xy, \\ 2xy = y^2 - 24. \end{cases}$

34. $\begin{cases} x + y + \sqrt{x + y} = 12, \\ x^3 + y^3 = 189. \end{cases}$

35. $\begin{cases} x^4 + x^2y^2 + y^4 = 133, \\ x^2 - xy + y^2 = 7. \end{cases}$

36. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$

37. $\begin{cases} 2x^2 - 5xy + 3x - 2y = 22, \\ 5xy + 7x - 8y - 2x^2 = 8. \end{cases}$

38. $\begin{cases} x + y = 74, \\ x^2 + y^2 = 3026. \end{cases}$

39. $\begin{cases} 7y^2 - 5x^2 + 20x + 13y = 29, \\ 5(x - 2)^2 - 7y^2 - 17y = -17. \end{cases}$

40. $\begin{cases}(3x+4y)(7x-2y)+3x+4y=44,\\(3x+4y)(7x-2y)-7x+2y=30.\end{cases}$

41. $\begin{cases}x+y=4,\\x^3+x^2y+xy^2+y^3=32.\end{cases}$

42. $\begin{cases}x^3-y^3=37,\\x-y=1.\end{cases}$

43. $\begin{cases}x^4+y^4=82,\\x+y=4.\end{cases}$

44. $\begin{cases}x^2+y^2-xy=80,\\x-y-xy=-8.\end{cases}$

45. $\begin{cases}8a+8b-ab-a^2=18,\\5a+5b-b^2-ab=24.\end{cases}$

46. $\begin{cases}(x^3+x^2y+xy^2+y^3)=120,\\x^3-x^2y+xy^2-y^3=40.\end{cases}$

47. $\begin{cases}2(x+4)^2-5(y-7)^2=75,\\7(x+4)^2+15(y-7)^2=1075.\end{cases}$

48. $\begin{cases}x^3+y^3=(a+b)(x-y),\\x^2-xy+y^2=a-b.\end{cases}$

HIGHER EQUATIONS INVOLVING QUADRATICS

151. An equation of a degree above the second may often be reduced to the solution of a quadratic after applying the factor theorem. See § 92.

Example. Solve $\quad 2x^3+x^2-10x+7=0. \qquad (1)$

By the factor theorem, $x-1$ is found to be a factor,

giving $\quad (x-1)(2x^2+3x-7)=0. \qquad (2)$

Hence by § 22, $x-1=0$ and $2x^2+3x-7=0. \qquad (3)$

From $x-1=0$, $\quad x=1. \qquad (4)$

From $2x^2+3x-7=0$, $\quad x=\dfrac{-3\pm\sqrt{65}}{4}. \qquad (5)$

Hence (4) and (5) give the three roots of (1).

EXERCISES

Solve each of the following equations:

1. $7x^3 - 11x^2 + 4x = 0.$
2. $3x^4 + x^3 + 2x^2 + 24x = 0.$
3. $3x^3 - 16x^2 + 23x - 6 = 0.$
4. $5x^3 + 2x^2 + 4x = -7.$
5. $28x^3 - 10x^2 - 44x = 6.$
6. $x^4 - 3x^3 + 3x^2 - x = 0.$
7. $4x^3 + 12x^2 - 3x - 9 = 0.$
8. $x^4 - 5x^3 + 2x^2 + 20x = 24.$
9. $6x^3 + 29x^2 - 19x = 16.$
10. $15x^4 + 49x^3 - 92x^2 + 28x = 0.$

EQUATIONS IN THE FORM OF QUADRATICS

152. If an equation of higher degree contains a certain expression and also the square of this expression, and involves the unknown in no other way, then the equation is a **quadratic in the given expression.**

Ex. 1. Solve $$x^4 + 7x^2 = 44. \qquad (1)$$

This may be written, $$(x^2)^2 + 7(x^2) = 44, \qquad (2)$$

which is a *quadratic in* x^2. Solving, we find
$$x^2 = 4 \text{ and } x^2 = -11. \qquad (3)$$

Hence, $$x = \pm 2 \text{ and } x = \pm\sqrt{-11}. \qquad (4)$$

Ex. 2. Solve $$x + 2 + 3\sqrt{x+2} = 18. \qquad (1)$$

Since $x + 2$ is the square of $\sqrt{x+2}$, this is a quadratic in $\sqrt{x+2}$.

Solving we find $$\sqrt{x+2} = 3 \text{ and } \sqrt{x+2} = -6. \qquad (2)$$

Hence $$x+2 = 9 \text{ and } x+2 = 36, \qquad (3)$$

Whence $$x = 7 \text{ and } x = 34. \qquad (4)$$

Ex. 3. Solve $(2x^2 - 1)^2 - 5(2x^2 - 1) - 14 = 0.$

First solve as a quadratic in $2x^2 - 1$ and then solve the two resulting quadratics in x.

Ex. 4. Solve $x^2 - 7x + 40 - 2\sqrt{x^2 - 7x + 69} = -26.$ (1)

Add 29 to each member, obtaining

$$x^2 - 7x + 69 - 2\sqrt{x^2 - 7x + 69} = 3. \qquad (2)$$

Solve (2) as a quadratic in $\sqrt{x^2 - 7x + 69}$, obtaining

$$\sqrt{x^2 - 7x + 69} = 3 \text{ and } \sqrt{x^2 - 7x + 69} = -1, \qquad (3)$$

whence $x^2 - 7x + 69 = 9$ or 1. (4)

The solution of the two quadratics in (4) will give the four values of x satisfying (1).

EXERCISES

Solve the following equations:

1. $x^6 + 2x^3 = 80.$
2. $5x - 4 - 2\sqrt{5x - 4} = 63.$
3. $(2 - x + x^2)^2 + x^2 - x = 18.$
4. $a^2 - 3a + 4 - 3\sqrt{a^2 - 3a + 4} = -2.$
5. $3a^6 - 7a^3 - 1998 = 0.$
6. $x^2 - 8x + 16 + 6\sqrt{x^2 - 8x + 16} = 40.$
7. $\left(a + \frac{2}{a}\right)^2 + 4\left(a + \frac{2}{a}\right) = 21.$
8. $a^8 - 97a^4 + 1296 = 0.$
9. $a^2 - 3a + 4 + \sqrt{a^2 - 3a + 15} = 19.$
10. $(5x - 7 + 3x^2)^2 + 3x^2 + 5x - 247 = 0.$
11. $\sqrt[3]{7x - 6} - 4\sqrt[6]{7x - 6} + 4 = 0.$

RELATIONS BETWEEN THE ROOTS AND THE COEFFICIENTS OF A QUADRATIC

153. If in the general quadratic, $ax^2 + bx + c = 0$, we divide both members by a and put $\frac{b}{a} = p$, $\frac{c}{a} = q$, we have $x^2 + px + q = 0$.

Solving, $x_1 = \dfrac{-p + \sqrt{p^2 - 4q}}{2}$, and $x^2 = \dfrac{-p - \sqrt{p^2 - 4q}}{2}$.

Adding x_1 and x_2, $$x_1 + x_2 = -\frac{2p}{2} = -p. \qquad (1)$$

Multiplying x_1 and x_2, $$x_1x_2 = \frac{p^2 - (p^2 - 4q)}{4} = q. \qquad (2)$$

Hence in a quadratic of the form $x^2 + px + q = 0$, the sum of the roots is $-p$, and the product of the roots is q.

The expression $p^2 - 4q = \dfrac{b^2}{a^2} - \dfrac{4c}{a} = \dfrac{b^2 - 4ac}{a^2}$.

Hence $p^2 - 4q$ is positive, negative, or zero, according as $b^2 - 4ac$ is positive, negative, or zero.

Hence, as found on pp. 387, 389, the roots of

$$ax^2 + bx + c = 0, \text{ or } x^2 + px + q = 0 \text{ are:}$$

real and distinct, if $b^2 - 4ac > 0$, *or* $p^2 - 4q > 0$, (3)

real and equal, if $b^2 - 4ac = 0$, *or* $p^2 - 4q = 0$, (4)

imaginary, if $b^2 - 4ac < 0$, *or* $p^2 - 4q < 0$. (5)

By means of (1) to (5), we may determine the character of the roots of a quadratic without solving it.

Ex. 1. Determine the character of the roots of

$$8x^2 - 3x - 9 = 0.$$

Since $b^2 - 4ac = 9 - 4 \cdot 8(-9) = 297 > 0$, the roots are real and distinct. Since $b^2 - 4ac$ is not a perfect square, the roots are irrational.

Since $q = -\frac{9}{8} = x_1x_2$, the roots have *opposite* signs.

Since $p = -\frac{3}{8}$ or $-p = \frac{3}{8} = x_1 + x_2$, the positive root is *greater* in absolute value.

Ex. 2. Examine $3x^2 + 5x + 2 = 0$.

Since $b^2 - 4ac = 25 - 4 \cdot 3 \cdot 2 = 1 > 0$, the roots are *real* and *distinct.*

Since $b^2 - 4ac$ is a perfect square, the roots are rational.

Since $q = \frac{2}{3} = x_1x_2$, the roots have the *same* sign.

Since $-p = -\frac{5}{3} = x_1 + x_2$, the roots are both *negative.*

Ex. 3. Examine $x^2 - 14x + 49 = 0$.

Since $p^2 - 4q = 196 - 4 \cdot 49 = 0$, the roots are real and coincident.

Ex. 4. Examine $x^2 - 7x + 15 = 0$.

Since $p^2 - 4q = 49 - 4 \cdot 15 = -11$, the roots are imaginary.

EXERCISES

Without solving, determine the character of the roots in each of the following:

1. $5x^2 - 4x - 5 = 0.$
2. $6x^2 + 4x + 2 = 0.$
3. $x^2 - 4x + 8 = 0.$
4. $2 + 2x^2 = 4x.$
5. $6x + 8x^2 = 9.$
6. $1 - a^2 = 3a.$
7. $6a - 30 = 3a^2.$
8. $6a^2 + 6 = 13a.$
9. $16m^2 + 4 = 16m.$
10. $25a^2 - 10a = 8.$
11. $20 - 13b - 15b^2 = 0.$
12. $10y^2 + 39y + 14 = 0.$
13. $3a^2 + 5a + 22.$
14. $3a^2 - 22a + 21 = 0.$
15. $5b^2 + 6b = 27.$
16. $6a - 17 = 11a^2.$

FORMATION OF EQUATIONS WHOSE ROOTS ARE GIVEN

154. Ex. 1. Form the equation whose roots are 7 and -4.

From (1) and (2), § 153, we have

$$x_1 + x_2 = -p = 7 + (-4) = 3. \quad \text{Hence } p = -3.$$

And $\quad x_1x_2 = q = 7(-4) = -28.$

Hence $\quad x^2 + px + q = 0$ becomes $x^2 - 3x - 28 = 0.$

In case the equation is to have more than two roots, we proceed as in the following example:

Ex. 2. Form the equation whose roots are 2, 3, and 5.

Recalling the solution by factoring, we may write the desired equation in the factored form as follows:

$$(x-2)(x-3)(x-5) = 0.$$

Obviously 2, 3, and 5, are the roots and the only roots of this equation. Hence the desired equation is:

$$(x-2)(x-3)(x-5) = x^3 - 10x^2 + 31x - 30 = 0.$$

EXERCISES

Form the equations whose roots are:

1. $3, -7$.
2. b, c.
3. $a, -b, -c$.
4. $5, -4, -2$.
5. $\sqrt{5}, -\sqrt{5}$.
6. $a-\sqrt{3}, a+\sqrt{3}$.
7. $-5, -6$.
8. $-b+k, -b-k$.
9. $\sqrt{-1}, -\sqrt{-1}$.
10. $a, -b$.
11. $8+\sqrt{3}, 8-\sqrt{3}$.
12. $2, 3, 4, 5$.
13. $3+2\sqrt{-1}, 3-2\sqrt{-1}$.
14. $5-\sqrt{-1}, 5+\sqrt{-1}$.
15. $1, \frac{1}{2}, \frac{1}{3}, 3$.
16. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}, \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

155. An expression of the second degree in a single letter may be **resolved into factors**, each of the first degree in that letter, by solving a quadratic equation.

Ex. 1. Factor $6x^2-17x+5$.

This trinomial may be written, $6(x^2-\frac{17}{6}x+\frac{5}{6})$.

Solving the equation, $x^2-\frac{17}{6}x+\frac{5}{6}=0$, we find $x_1=\frac{1}{3}$ and $x_2=\frac{5}{2}$. Hence by the factor theorem, § 92, $x-\frac{1}{3}$ and $x-\frac{5}{2}$ are factors of $x^2-\frac{17}{6}x+\frac{5}{6}$. And finally

$$\begin{aligned}6(x^2-17x+5)&=6(x-\tfrac{1}{3})(x-\tfrac{5}{2})=3(x-\tfrac{1}{3})\cdot 2(x-\tfrac{5}{2})\\&=(3x-1)(2x-5).\end{aligned}$$

This process is not needed when the factors are *rational*, but it is applicable equally well when the factors are *irrational* or *imaginary*.

Ex. 2. Factor $3x^2+8x-7=3(x^2+\frac{8}{3}x-\frac{7}{3})$.

Solving the equation $x^2+\frac{8}{3}x-\frac{7}{3}=0$, we find,

$$x_1=\frac{-4+\sqrt{37}}{3} \text{ and } x_2=\frac{-4-\sqrt{37}}{3}.$$

Hence as above:

$$\begin{aligned}3x^2+8x-7&=3\left[x-\frac{-4+\sqrt{37}}{3}\right]\left[x-\frac{-4-\sqrt{37}}{3}\right]\\&=3\left[x+\frac{4}{3}-\frac{\sqrt{37}}{3}\right]\left[x+\frac{4}{3}+\frac{\sqrt{37}}{3}\right].\end{aligned}$$

EXERCISES

In exercises 1 to 16, p. 410, transpose all terms of each equation to the first member, and then factor this member.

PROBLEMS INVOLVING QUADRATIC EQUATIONS

In each of the following problems, interpret both solutions of the quadratic involved:

1. The area of a rectangle is 2400 square feet and its perimeter is 200 feet. Find the length of its sides.

2. The area of a rectangle is a square feet and its perimeter is $2b$ feet. Find the length of its sides. Solve 1 by substitution in the formula thus obtained.

3. A picture measured inside the frame is 18 by 24 inches. The area of the frame is 288 square inches. Find its width.

4. If in problem 3 the sides of the picture are a and b and the area of the frame c, find the width of the frame.

5. The sides a and b of a right triangle are increased by the same amount, thereby increasing the square on the hypotenuse by $2k$. Find by how much each side is increased.

Make a problem which is a special case of this and solve it by substitution in the formula just obtained.

6. The hypotenuse c and one side a are each increased by the same amount, thereby increasing the square on the other side by $2k$. Find how much was added to the hypotenuse.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

7. A rectangular park is 80 by 120 rods. Two driveways of equal width, one parallel to the longer and one to the shorter side, run through the park. What is the width of the driveways if their combined area is 591 square rods?

8. If in problem 7 the park is a rods wide and b rods long and the area of the driveways is c square rods, find their width.

9. The diagonal of a rectangle is a and its perimeter $2b$. Find its sides.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

10. If in problem 9 the difference between the length and width is b and the diagonal is a, find the sides. Show how one solution can be made to give the results for both problems 9 and 10.

11. Find two consecutive integers whose product is a.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

What special property must a have in order that this problem may be possible. Answer this from the formula.

12. A rectangular sheet of tin, 12 by 16 inches, is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cubic inches.

The resulting equation is of the third degree. Solve it by factoring. See § 151. Obtain three results and determine which are applicable to the problem.

13. A square piece of tin is made into an open box containing a cubic inches, by cutting from each corner a square whose side is b inches and then turning up the sides. Find the dimensions of the original piece of tin.

14. A rectangular piece of tin is a inches longer than it is wide. By cutting from each corner a square whose side is b inches and turning up the sides, an open box containing c cubic inches is formed. Find the dimensions of the original piece of tin.

15. The hypotenuse of a right triangle is 20 inches longer than one side and 10 inches longer than the other. Find the dimensions of the triangle.

16. If in problem 15 the hypotenuse is a inches longer than one side and b inches longer than the other, find the dimensions of the triangle.

17. The area of a circle exceeds that of a square by 10 square inches, while the perimeter of the circle is 4 less than that of the square. Find the side of the square and the radius of the circle.

Use $3\frac{1}{7}$ as the value of π.

18. If in problem 17 the area of the circle exceeds that of the square by a square inches, while its perimeter is $2b$ inches less than that of the square, find the dimensions of the square and the circle.

Determine from this general solution under what conditions the problem is possible.

19. Find three consecutive integers such that the sum of their squares is a.

Make a problem which is a special case of this and solve it by means of the formula just obtained. From the formula discuss the cases, $a = 2$, $a = 5$, $a = 14$. Find another value of a for which the problem is possible.

20. The difference of the cubes of two consecutive integers is 397. Find the integers.

21. The upper base of a trapezoid is 8 and the lower base is 3 times the altitude. Find the altitude and the lower base if the area is 78.

See problem 7, p. 348.

22. The lower base of a trapezoid is 4 greater than twice the altitude, and the upper base is $\frac{1}{2}$ the lower base. Find the two bases and the altitude if the area is $52\frac{1}{2}$.

23. The lower base of a trapezoid is twice the upper, and its area is 72. If $\frac{1}{2}$ the altitude is added to the upper base, and the lower is increased by $\frac{1}{4}$ of itself, the area is then 120. Find the dimensions of the trapezoid.

24. The upper base of a trapezoid is equal to the altitude, and the area is 48. If the altitude is decreased by 4, and the upper base by 2, the area is then 14. Find the dimensions of the trapezoid.

25. The upper base of a trapezoid is 4 more than $\frac{1}{2}$ the lower base, and the area is 84. If the upper base is decreased by 5, and the lower is increased by $\frac{1}{2}$ the altitude, the area is 78. Find the dimensions of the trapezoid.

26. The area of an equilateral triangle multiplied by $\sqrt{3}$, plus 3 times its perimeter, equals 81. Find the side of the triangle.

See problem 15, p. 236, E. C.

27. The area of a regular hexagon multiplied by $\sqrt{3}$, minus twice its perimeter, is 504. Find the length of its side.

See problem 20, p. 237, E. C.

28. If a times the perimeter of a regular hexagon, plus $\sqrt{3}$ times its area, equals b, find its side.

29. The perimeter of a circle divided by π, plus $\sqrt{3}$ times the area of the inscribed regular hexagon, equals $78\frac{3}{4}$. Find the radius of the circle.

30. The area of a regular hexagon inscribed in a circle plus the perimeter of the circle is a. Find the radius of the circle.

31. One edge of a rectangular box is increased 6 inches, another 3 inches, and the third is decreased 4 inches, making a cube whose volume is 864 cubic inches greater than that of the original box. Find its dimensions.

32. Of two trains one runs 12 miles per hour faster than the other, and covers 144 miles in one hour less time. Find the speed of each train.

In a township the main roads run along the section lines, one half of the road on each side of the line.

33. Find the area included by the main roads of a township if they are 4 rods wide.

34. If the area included by the main roads of a township is 11,196 square rods, find the width of the roads.

35. Find the width of the roads in problem 34 if the area included by them is a square rods.

CHAPTER VIII

ALGEBRAIC FRACTIONS

156. An **algebraic fraction** is the indicated quotient of two algebraic expressions.

Thus $\frac{n}{d}$ means n *divided by* d.

From the definition of a fraction and § 11, it follows that the *product of a fraction and its denominator equals its numerator.*

That is,
$$d \cdot \frac{n}{d} = n.$$

REDUCTION OF FRACTIONS

157. The **form** of a fraction may be modified in various ways without changing its **value**. Any such transformation is called **a reduction of the fraction.**

The most important reductions are the following:

(*A*) *By manipulation of signs.*

E.g. $\frac{n}{d} = -\frac{-n}{d} = -\frac{n}{-d} = \frac{-n}{-d}$; $\frac{b-a}{c-d} = -\frac{a-b}{c-d} = \frac{a-b}{d-c}$.

(*B*) *To lowest terms.*

E.g.
$$\frac{x^4+x^2+1}{x^6-1} = \frac{(x^2+x+1)(x^2-x+1)}{(x-1)(x^2+x+1)(x+1)(x^2-x+1)}$$
$$= \frac{1}{(x-1)(x+1)}.$$

(*C*) *To integral or mixed expressions.*

E.g. $\frac{2x^3+x^2+x+2}{x^2+1} = 2x+1+\frac{-x+1}{x^2+1} = 2x+1-\frac{x-1}{x^2+1}$.

(*D*) *To equivalent fractions having a common denominator.*

E.g. $\frac{2}{x+3}$ and $\frac{3}{x+2}$ become respectively $\frac{2(x+2)}{(x+3)(x+2)}$ and $\frac{3(x+3)}{(x+3)(x+2)}$; $a+1$ and $\frac{1}{a-1}$ become respectively $\frac{a^2-1}{a-1}$ and $\frac{1}{a-1}$.

158. These reductions are useful in connection with the various operations upon fractions. They depend upon the principles indicated below.

Reduction (*A*) is simply an application of the law of signs in division, § 28. It is often needed in connection with reduction (*D*). See § 159.

Reduction (*B*) depends upon the theorem, § 47, $\frac{ak}{bk}=\frac{a}{b}$, by which *a common factor may be removed from both terms of a fraction.* It is useful in keeping expressions *simplified.* This reduction is *complete* when numerator and denominator have been divided by their H. C. F. See §§ 95–102.

Reduction (*C*) is merely the process of performing the indicated division, the result being *integral* when the division is *exact,* otherwise a *mixed expression.*

In case there is a *remainder* after the division has been carried as far as possible, this part of the quotient can only be *indicated.*

Thus
$$\frac{D}{d}=q+\frac{R}{d},$$

in which D is dividend, d is divisor, q is quotient, and R is remainder.

Reduction (*D*) depends upon the theorem of § 47, $\frac{a}{b}=\frac{ka}{kb}$, by which *a common factor is introduced into the terms of a fraction.*

A fraction is thus reduced to another fraction whose denominator is *any required multiple* of the given denominator.

If two or more fractions are to be reduced to equivalent fractions having a common denominator, this denominator must be a *common multiple* of the given denominators, and for simplicity the L. C. M. is used.

EXERCISES

Reduce the following so that the letters in each factor shall occur in alphabetical order, and no negative sign shall stand before a numerator or denominator, or before the first term of any factor.

1. $\frac{n-m}{b-a}$.

2. $-\frac{(b-a)(c-d)}{x(s-r-t)}$.

3. $\frac{-(x-y)}{(b-a)(c-d)}$.

4. $\frac{-(x-y)(z-y)}{-(b-a)(c-d)}$.

5. $\frac{r-s}{(a-b)(c-b)(c-a)}$.

6. $\frac{-a(c+b)}{b(c-a)}$.

7. $\frac{-(c-a)(d-c)}{(a-b)(b-c)}$.

8. $\frac{(b-a)(c-b)(c-a)}{(y-x)(y-z)(z-x)}$.

9. $-\frac{-1}{(a-b)(b-c)(c-a)}$.

10. $\frac{(c-b-a)(b-a-c)}{3(a-c)(b-c)(c-a)}$.

11. $\frac{(3c-2a)(4b-a)d}{(-a+b)(a-b)(c-a)}$.

12. $\frac{-(-r-s)(s-t)(t-r)}{(n-m)(-k-m-l)}$.

Reduce each of the following to lowest terms:

13. $\frac{a^4-b^4}{a^6-b^6}$.

14. $\frac{c^2-(a-b)^2}{(a+c)^2-b^2}$.

15. $\frac{7ax^2-56a^4x^5}{28x^2(1-64a^6x^6)}$.

16. $\frac{m^3+5m^2+7m+3}{m^2+4m+3}$.

17. $\frac{a^3-7a+6}{a^3-7a^2+14a-8}$.

18. $\frac{x^3+2x^2+2x+1}{x^4+x^3-x^2-2x-2}$.

19. $\frac{2x^3-x^2-8x-3}{2x^3-3x^2-7x+3}$.

20. $\frac{4x^3+8x^2-3x+5}{6x^3-5x^2+4x-1}$.

21. $\frac{x^2-xy+y^2+x-y+3}{x^3+y^3+x^2-y^2+3x+3y}$.

22. $\frac{a^4+a^2b^2+b^4+a^3+b^3}{a^2+ab+b^2+a+b}$.

23. $\dfrac{x^4+4x^3y+6x^2y^2+4xy^3+y^4-a^4}{x^2+2xy+y^2-a^2}$.

24. $\dfrac{x^2y-x^2z+y^2z-xy^2+xz^2-yz^2}{x^2-(y+z)x+yz}$.

25. $\dfrac{2x^4-x^3-20x^2+16x-3}{3x^4+5x^3-30x^2-41x+15}$.

26. $\dfrac{3a^3-8a^2b-5ab^2+6b^3}{a^3+a^2b-9ab^2-9b^3}$.

27. $\dfrac{2r^3+r^2s+rs^2+2s^3}{2r^4+r^3s+3r^2s^2+rs^3+2s^4}$.

Reduce each of the following to an integral or mixed expression:

28. $\dfrac{x^4+1}{x+1}$.

29. $\dfrac{x^5+1}{x+1}$.

30. $\dfrac{x^4}{x-1}$.

31. $\dfrac{a^3}{a^2+a+1}$.

32. $\dfrac{c^5}{c^3+c^2-c+1}$.

33. $\dfrac{x^2-x+1}{x^2+x+1}$.

34. $\dfrac{a^4+a^2b^2+b^4}{a-b}$.

35. $\dfrac{3a^3-3a^2+3a-1}{a-2}$.

36. $\dfrac{x^3-x^2-x+1}{x^3+x^2+x-1}$.

37. $\dfrac{4m^4-3m^3+3}{2m^2-2m+1}$.

Reduce each of the following sets of expressions to equivalent fractions having the lowest common denominator:

38. $\dfrac{1}{x^4-3x^2y^2+y^4}$, $\dfrac{1}{x^2-xy-y^2}$, $\dfrac{1}{x^2+xy-y^2}$.

39. $\dfrac{a+b}{5a^2c+12cd-6ad-10ac^2}$, $\dfrac{a}{5ac-6d}$, $\dfrac{b}{a-2c}$.

40. $\dfrac{x^2+y^2}{x^3+y^3+x^2-xy+y^2}$, $\dfrac{x+y-1}{x^2-xy+y^2}$, $\dfrac{x^2+xy+y^2}{x+y+1}$.

41. $\dfrac{x}{(a-b)(c-b)(c-a)}$, $\dfrac{y}{(a-b)(b-c)(a-c)}$, $\left[\dfrac{z}{(b-a)(b-c)(a-c)}\right.$.

42. $\dfrac{a}{b+c}$, $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$, d.

43. $\dfrac{b-c}{(a-c)(a-b)}$, $\dfrac{a-b}{(c-a)(b-c)}$, $\dfrac{c-a}{(b-a)(c-b)}$.

44. $\dfrac{m-n}{a^3-6a^2+11a-6}$, $\dfrac{a+2}{a^2-4a+3}$, $\dfrac{a+3}{a^2-3a+2}$.

If a, b, m are positive numbers, arrange each of the following sets in decreasing order. Verify the results by substituting convenient Arabic numbers for a, b, m.

Suggestion. Reduce the fractions in each set to equivalent fractions having a common denominator.

45. $\dfrac{a}{a+1}$, $\dfrac{2a}{a+2}$, $\dfrac{3a}{a+3}$. 46. $\dfrac{m}{2m+1}$, $\dfrac{2m}{3m+2}$, $\dfrac{3m}{4m+3}$.

47. $\dfrac{a+3b}{a+4b}$, $\dfrac{a+b}{a+2b}$, $\dfrac{a+4b}{a+5b}$.

48. Show that, for a different from zero, neither $\dfrac{n+a}{d+a}$ nor $\dfrac{n-a}{d-a}$ can equal $\dfrac{n}{d}$, unless $n=d$. State this result in words, and fix it in mind as an impossible reduction of a fraction.

ADDITION AND SUBTRACTION OF FRACTIONS

159. Fractions which have a common denominator are added or subtracted in accordance with the distributive law for division, §§ 30, 31.

That is, $$\frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d}.$$

In order to add or subtract fractions not having a common denominator, they should first be reduced to equivalent fractions having a common denominator.

When several fractions are to be combined, it is sometimes best to take only part of them at a time. In any case it is advantageous to keep all expressions in the factored form as long as possible.

Ex. $$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}+\frac{1}{(3-x)(4-x)}.$$

Taking the first two together, we have

$$\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}=\frac{2x-4}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)(x-3)}.$$

Taking this result with the third,

$$\frac{2}{(x-1)(x-3)}+\frac{1}{(x-3)(x-4)}=\frac{3x-9}{(x-1)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

If all are taken at once, the work should be carried out as follows: The numerator of the sum is

$$(x-3)(x-4)+(x-1)(x-4)+(x-1)(x-2).$$

Adding the first two terms with respect to $(x-4)$, we have

$$2(x-2)(x-4)+(x-1)(x-2)\cdot$$

Adding these with respect to $(x-2)$, we have $3(x-3)(x-2)$.

Hence the sum is $\dfrac{3(x-3)(x-2)}{(x-1)(x-2)(x-3)(x-4)}=\dfrac{3}{(x-1)(x-4)}.$

EXERCISES

Perform the following indicated additions and subtractions:

1. $\dfrac{2}{x-3}+\dfrac{3}{x-4}-\dfrac{4}{x-5}.$

2. $\dfrac{3}{4(x+3)}-\dfrac{5}{8(x+5)}-\dfrac{1}{8(x+1)}.$

3. $\dfrac{1}{2(x-1)}-\dfrac{4}{x-2}+\dfrac{7}{2(x-3)}.$

4. $\dfrac{1}{12(x+1)}-\dfrac{7}{3(x-2)}+\dfrac{13}{4(x-3)}.$

5. $\dfrac{2}{(x+1)^2}+\dfrac{3}{x+1}+\dfrac{4}{x-2}$.

6. $\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$.

7. $\dfrac{5x+6}{x^2+x+1}-\dfrac{3x-4}{x^2-x+1}$.

8. $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$.

9. $\dfrac{2}{(x-2)^2}-\dfrac{1}{x-2}+\dfrac{1}{x+1}$.

10. $\dfrac{2}{(x-2)^2}+\dfrac{1}{5(x-2)}-\dfrac{x+2}{5(x^2+1)}$.

11. $\dfrac{1}{(x-1)^2}+\dfrac{1}{(x-1)}-\dfrac{1}{(x^2-1)}$.

12. $\dfrac{1}{2(1-3x)^3}+\dfrac{3}{8(1-3x)^2}+\dfrac{3}{32(1-3x)}+\dfrac{1}{32(1+x)}$.

13. $\dfrac{1}{(1-a)(2-a)}-\dfrac{1}{(2-a)(a-3)}+\dfrac{2}{(3-a)(a-1)}$.

14. $\dfrac{xy}{(z-y)(x-z)}-\dfrac{yz}{(x-z)(x-y)}-\dfrac{xz}{(y-x)(y-z)}$.

15. $\dfrac{1}{a-1}-\dfrac{2a-5}{a^2-2a+1}-\dfrac{5a^2-3a-2}{(a-1)^3}$.

16. $\dfrac{1}{m^2+m+1}-\dfrac{1}{m^2-m+1}+\dfrac{2m+2}{m^4+m^2+1}$.

17. $\dfrac{1}{b^2-3b+2}+\dfrac{1}{b^2-5b+6}-\dfrac{2}{b^2-4b+3}$.

18. $\dfrac{r+s}{(r-t)(s-t)}-\dfrac{s+t}{(r-s)(t-r)}-\dfrac{r+t}{(t-s)(s-r)}$.

19. $\dfrac{p^2+q^2}{(p-q)(p+r)}+\dfrac{q^2-pr}{(q-r)(q-p)}+\dfrac{r^2+pq}{(r-q)(r+p)}$.

20. $\dfrac{3x^2+1}{5x^2-18x+9}-\dfrac{2x^2+2}{4x^2-11x-3}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

160. *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

That is, $$\frac{a}{b}\cdot\frac{n}{d}=\frac{an}{bd}.$$

For let $$x=\frac{a}{b}\cdot\frac{n}{d}.$$

Then $$bdx=bd\left(\frac{a}{b}\cdot\frac{n}{d}\right). \quad \S\ 7$$

$$bdx=b\cdot\frac{a}{b}\cdot d\cdot\frac{n}{d}. \quad \S\ 8$$

$$bdx=an. \quad \S\ 11$$

Hence, $$x=\frac{an}{bd}.$$

Therefore, $$\frac{a}{b}\cdot\frac{n}{d}=\frac{an}{bd}. \quad \S\ 2$$

It follows that a fraction is raised to any power by raising numerator and denominator separately to that power.

For $\frac{a}{b}\cdot\frac{a}{b}=\frac{a^2}{b^2}$, $\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}=\frac{a^3}{b^3}$, etc.

A fraction multiplied by itself inverted equals $+1$.

For $\frac{n}{d}\cdot\frac{d}{n}=\frac{nd}{nd}=+1$ and $-\frac{n}{d}\cdot\left(-\frac{d}{n}\right)=\frac{nd}{nd}=+1$.

161. Definitions. If the product of two numbers is 1, each is called the **reciprocal** of the other. Hence by § 160, the reciprocal of a fraction is the fraction inverted.

Also, since from $ab=1$ we have $a=\frac{1}{b}$ and $b=\frac{1}{a}$, it follows that if two numbers are reciprocals of each other, then either one is the quotient obtained by dividing 1 by the other.

162. *To divide by any number is equivalent to multiplying by its reciprocal.*

For it is an immediate consequence of § 29 that

$$n \div d \text{ or } \frac{n}{d} = n \cdot \frac{1}{d}.$$

To divide a number by a fraction is equivalent to multiplying by the fraction inverted.

For by § 161 the reciprocal of the fraction is the fraction inverted.

A fraction is divided by an integer by multiplying its denominator or dividing its numerator by that integer.

For $\qquad \frac{n}{d} \div a = \frac{n}{d} \cdot \frac{1}{a} = \frac{n}{ad},$

and $\qquad \frac{n}{d} \div a = \frac{n \div a}{d}$, since $\frac{n}{ad} = \frac{n \div a}{d}$ by § 47.

In multiplying and dividing fractions their terms should at once be put into *factored* forms.

When mixed expressions or sums of fractions are to be multiplied or divided, these operations are indicated by means of parentheses, and the additions or subtractions within the parentheses should be performed first, § 38.

Ex. Simplify $\left[\left(1 - a + \frac{2a^2}{1+a}\right) \div \left(\frac{1}{1+a} - \frac{1}{1-a}\right)\right] \cdot \frac{3a^3}{a^4 - 1}.$

Performing the indicated operations within the parentheses, we have

$$\left[\frac{1+a^2}{1+a} \div \frac{2a}{a^2-1}\right] \cdot \frac{3a^3}{a^4-1} = \frac{1+a^2}{1+a} \cdot \frac{a^2-1}{2a} \cdot \frac{3a^3}{(a^2-1)(a^2+1)} = \frac{3a^2}{2(a+1)}.$$

EXERCISES

Perform the following indicated operations and reduce each result to its simplest form.

1. $\frac{x^4 + x^2y^2 + y^4}{x^3 - y^3} \cdot \frac{x^2 - y^2}{x^3 + y^3}.$

2. $\dfrac{a^2 - b^2x^2 + acx^2 - bcx^3}{36\,a^4 - 9\,a^2 + 24\,a - 16} \div \dfrac{-ay^2 - bxy^2 - cx^2y^2}{20\,x^2 - 15\,ax^2 - 30\,a^2x^2}.$

3. $\dfrac{20\,r^2s^2 + 23\,rst - 21\,t^2}{8\,m^2n^3 - 48\,m^2n^2y + 72\,m^2ny^2} \times \dfrac{12\,mn^3 - 28\,mn^2y - 24\,mny^2}{10\,r^2s^2 + 24\,rst - 18\,t^2}$ $\left[\div \dfrac{3\,n + 2\,y}{2\,s + 3\,t}.\right.$

4. $\left(\dfrac{a}{b} - \dfrac{b}{a}\right)\left(\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} - \dfrac{a}{b} - \dfrac{b}{a} + 1\right) \div \dfrac{a^5 + b^5}{a - b}.$

5. $\left(\dfrac{1}{x} + \dfrac{1}{y}\right)\left(\dfrac{1}{x} - \dfrac{1}{y}\right)\left(1 - \dfrac{x - y}{x + y}\right)\left(2 + \dfrac{2\,y}{x - y}\right).$

6. $\left(\dfrac{a}{a - b} - \dfrac{b}{a + b}\right)\left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right) \div \left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right).$

7. $\left(1 + \dfrac{b}{a - b}\right)\left(1 - \dfrac{b}{a + b}\right) \div \left(1 + \dfrac{b^2}{a^2 - b^2}\right).$

8. $\left(\dfrac{m + n}{m - n} - \dfrac{m - n}{m + n}\right)\left(m + n + \dfrac{2\,n^2}{m - n}\right) \div \left(\dfrac{m + n}{m - n} + \dfrac{m - n}{m + n}\right).$

9. $\left(\dfrac{x^2 + y^2}{x^2 - y^2} - \dfrac{x^2 - y^2}{x^2 + y^2}\right) \cdot \left(x^2 + y^2 + \dfrac{2\,x^2y^2 + 2\,y^4}{x^2 - y^2}\right) \div \left(\dfrac{x + y}{x - y} + \dfrac{x - y}{x + y}\right).$

10. $\left(\dfrac{x + y + z}{x + y} + \dfrac{z^2}{(x + y)^2}\right) \cdot \left(\dfrac{(x + y)^3}{(x + y)^3 - z^3}\right) \cdot \left(1 + \dfrac{z}{x + y}\right).$

11. $\dfrac{a^2 + ab + b^2}{a^2 - ab + b^2} \cdot \dfrac{a + b}{a^3 - b^3} \cdot \left(a^2 + \dfrac{b^3 - a^2b}{a + b}\right).$

12. $\dfrac{m^2 + mn}{m^2 + n^2} \cdot \dfrac{m^3 - mn^2 - m^2 + n^2}{m^3n - n^4} \cdot \dfrac{m^2n^2 + mn^3 + n^4}{m^4 - 2\,m^3 + m^2}$

$$\div \frac{m^3n + 2\,m^2n^2 + mn}{m^4 - n^4}.$$

13. $\left(xy^9 + x^9y - \dfrac{2\,x^{11}y^3}{x^2y^2}\right) \div \left[\dfrac{x^2 + y^2}{x^2} \cdot \left(\dfrac{1}{y^2} - \dfrac{1}{z^2}\right) - \dfrac{x^2 + y^2}{y^2} \cdot \left(\dfrac{1}{x^2} - \dfrac{1}{z^2}\right)\right].$

COMPLEX FRACTIONS

163. A fraction which contains a fraction either in its numerator or in its denominator or in both is called a **complex fraction.**

Since every fraction is an indicated operation in division, any complex fraction may be simplified by performing the indicated division.

It is usually better, however, to remove all the minor denominators at once by multiplying both terms of the complex fraction by the least common multiple of all the minor denominators according to § 47.

For example, $$\frac{\frac{x}{3}+\frac{x}{2}}{\frac{2x^2}{3}-\frac{3}{2}}=\frac{\left(\frac{x}{3}+\frac{x}{2}\right)\cdot 6}{\left(\frac{2x^2}{3}-\frac{3}{2}\right)\cdot 6}=\frac{2x+3x}{4x^2-9}=\frac{5x}{4x^2-9}.$$

A complex fraction may contain another complex fraction in one of its terms.

E.g. $\dfrac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}}$ has the complex fraction $\dfrac{a+1}{a+\dfrac{1}{a-1}}$

in its denominator. This latter fraction is first reduced by multiplying its numerator and denominator by $a-1$, giving

$$\frac{a+1}{a+\dfrac{1}{a-1}}=\frac{a^2-1}{a^2-a+1}.$$

Substituting this result in the given fraction, we have

$$\frac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}}=\frac{1}{a+\dfrac{a^2-1}{a^2-a+1}}=\frac{a^2-a+1}{a^3+a-1}.$$

EXERCISES

Simplify each of the following,

1. $$\frac{\dfrac{m^2+mn}{m^2-n^2}}{\dfrac{m}{m-n}-\dfrac{n}{m+n}}.$$

2. $$\frac{\dfrac{a^4-b^4}{a^2-2\,ab+b^2}}{\dfrac{a^2+ab}{a-b}}.$$

3. $$\frac{\dfrac{x^5-3\,x^4y+3\,x^3y^2-x^2y^3}{x^3y-y^4}}{\dfrac{x^5-2\,x^4y+x^3y^2}{x^2y^2+xy^3+y^4}}.$$

4. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2\,a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2\,a}{a^2-x^2}}.$$

5. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2\,a}{a^2+x^2}}{\dfrac{1}{a-x}-\dfrac{1}{a+x}+\dfrac{2\,x}{a^2+x^2}}.$$

6. $$\frac{m^2-mn+n^2-\dfrac{m^3-n^3}{m+n}}{m^2+mn+n^2+\dfrac{m^3+n^3}{m-n}}.$$

7. $$\frac{\dfrac{a-\dfrac{1}{a^2}}{a-2+\dfrac{1}{a}}}{\dfrac{a^2+1+\dfrac{1}{a^2}}{a-2+\dfrac{2}{a}-\dfrac{1}{a^2}}}.$$

8. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}.$$

9. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}$$

10. $$\frac{3}{3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{x}}}}.$$

EQUATIONS INVOLVING ALGEBRAIC FRACTIONS

164. In solving a fractional equation, it is usually convenient to **clear it of fractions**, that is, to transform it into an equivalent equation containing no fractions.

In case no denominator contains any unknown this may be done by multiplying both members by the L. C. M. of all the denominators, § 62.

When, however, the unknown appears in any denominator, multiplying by the L. C. M. of all the denominators *may or may not* introduce new roots, as shown in the following examples.

It may easily be shown, that multiplying an *integral* equation by any expression containing the unknown *always* introduces new roots.

Ex. 1. Solve $$\frac{2}{x-2}+\frac{1}{x-3}=2. \qquad (1)$$

Clearing of fractions by multiplying by $(x-2)(x-3)$, and simplifying, we have
$$2x^2-13x+20=(x-4)(2x-5)=0. \qquad (2)$$

The roots of (2) are 4 and $2\frac{1}{2}$, both of which satisfy (1). Hence no new root was introduced by clearing of fractions.

Ex. 2. Solve $$\frac{1}{x-1}=\frac{1}{(x-1)(x-2)}. \qquad (1)$$

Clearing of fractions, we have,
$$x-2=1. \qquad (2)$$

The only root of (2) is $x=3$, which is also the only root of (1). Hence no new root was introduced.

Ex. 3. Solve $$\frac{x-2}{x^2-4}=1. \qquad (1)$$

Clearing of fractions and simplifying, we have,
$$x^2-x-2=(x-2)(x+1)=0. \qquad (2)$$

The roots of (2) are 2 and -1. Now $x=-1$ is a root of (1), but $x=2$ is *not*, since we are not permitted to make a substitution which reduces a denominator to zero, § 50. Hence a new root has been introduced and (1) and (2) are not equivalent.

If the fraction $\frac{x-2}{x^2-4}$ is first reduced to lowest terms, we have the equation
$$\frac{1}{x+2}=1. \qquad (3)$$

Clearing of fractions, $$x+2=1. \qquad (4)$$

Now (3) and (4) are equivalent, -1 being the only root of each.

Ex. 4. Solve $$\frac{4x}{x^2-1}-\frac{x+1}{x-1}=1. \qquad (1)$$

Clearing of fractions and simplifying,

$$x^2 - x = x(x-1) = 0. \qquad (2)$$

The roots of (2) are 0 and 1. $x = 0$ satisfies (1), but $x = 1$ does not, since it is not a permissible substitution in either fraction of (1). Hence a new root has been introduced.

165. Examples 3 and 4 illustrate the *only cases in which new roots can be introduced* by multiplying by the L. C. M. of the denominators.

This can be shown by proving certain important theorems, the results of which are here used in the following directions for solving fractional equations:

(1) Reduce all fractions to their lowest terms.

(2) Multiply both members by the least common multiple of the denominators.

(3) Reject any root of the *resulting* equation which reduces any denominator of the *given* equation to zero. The remaining roots will then satisfy both equations, and hence are the solutions desired.

If when each fraction is in its lowest terms the given equation contains no two which have a factor common to their denominators, then *no new root* can enter the resulting equation and none need to be rejected. See Ex. 1 and Ex. 3 after being reduced.

If, however, any two or more denominators have some common factor $x - a$, then $x = a$ *may or may not be a new root* in the resulting equation, but in any case it is the only possible *kind* of new root which can enter, and must be tested. Compare Exs. 2 and 4.

Ex. 5. Examine $$\frac{3x+7}{x^2+2x+11}+\frac{5x}{x^2+3x+2}-\frac{x+1}{x-1}=8.$$

Since each fraction is in its lowest terms and no two denominators contain a common factor, then clearing of fractions will give an equation equivalent to the given one.

Ex. 6. Examine $\dfrac{2x+3}{x^2+5x+6}-\dfrac{x-7}{x^2+3x+2}=4.$

Each fraction is in its lowest terms, but the two denominators have the factor $x+2$ in common. Hence $x=-2$ is the *only possible* new root which can enter the resulting integral equation, but on trial it is found not to be a root. Hence the two equations are equivalent.

EXERCISES

Determine whether each of the following when cleared of fractions produces an equivalent equation, and solve each.

1. $\dfrac{3x^3+3}{3x^2-7x+3}=x-7.$

2. $\dfrac{x^2+4x+4}{x^2-4}=2x+3.$

3. $\dfrac{3}{2x^2-x-1}+\dfrac{5}{x^2-1}+\dfrac{1}{x+1}=0.$

4. $\dfrac{2x}{2x-1}+\dfrac{x}{x+1}-\dfrac{3x}{x-1}=-1.$

5. $\dfrac{1}{3(x-1)}-\dfrac{1}{x^2-1}=\dfrac{1}{4}.$

6. $\dfrac{2a-1}{a}+\dfrac{1}{2}=\dfrac{3a}{3a-1}.$

7. $\dfrac{2}{x-a}+\dfrac{3}{x-b}=\dfrac{6}{x-c}.$

8. $\dfrac{1}{a-x}-\dfrac{1}{a+x}=-\dfrac{3+x^2}{a^2-x^2}.$

9. $a^2b-\dfrac{a+x}{b}=ab^2-\dfrac{b+x}{a}.$

10. $b=\dfrac{x-a}{1-ax}.$

11. $\dfrac{2}{x-10}+10=x+\dfrac{2}{10-x}.$

12. $\dfrac{6-x}{x-4}+\dfrac{x-4}{6-x}=\dfrac{c}{d}.$

13. $\dfrac{a}{2a-1}+\dfrac{24}{4a^2-1}=\dfrac{2(a-4)}{2a+1}-\dfrac{1}{9}.$

14. $\dfrac{a}{a-1}+\dfrac{a-1}{a}=\dfrac{a^2+a-1}{a^2-a}.$

15. $\dfrac{1}{a^2-4}-\dfrac{3}{2-a}=1+\dfrac{1}{3(a+2)}.$

16. $\dfrac{ax+b}{a+bx}+\dfrac{cx+d}{c+dx}=\dfrac{ax-b}{a-bx}+\dfrac{cx-d}{c-dx}.$

17. $\dfrac{(a-x)(x-b)}{(a-x)-(x-b)}=x.$ 18. $\dfrac{x+m-2n}{x+m+2n}=\dfrac{n+2m-2x}{n-2m+2x}.$

19. $\dfrac{(a-x)^2-(x-b)^2}{(a-x)(x-b)}=\dfrac{4ab}{a^2-b^2}.$

20. $\dfrac{1+3x}{5+7x}-\dfrac{9-11x}{5-7x}=14\cdot\dfrac{(2x-3)^2}{25-49x^2}$

21. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}=\dfrac{4ab}{4b^2-x^2}.$

22. $\dfrac{1}{x-2}+\dfrac{7x}{24(x+2)}=\dfrac{5}{x^2-4}.$

23. $\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}=\dfrac{2(a^2+1)}{(1+a)(1-a)}.$

24. $\dfrac{x-m}{x+m}=\dfrac{n-x}{n+x}.$ 25. $\dfrac{1}{x-a}-\dfrac{2a}{x^2-a^2}=b.$

26. $\dfrac{4}{3x+1}+\dfrac{4(3x-1)}{2x+1}=\dfrac{2x+1}{3x+1}.$

27. $\dfrac{2x+3}{2(2x-1)}+\dfrac{7-3x}{3x-4}+\dfrac{x-7}{2(x+1)}=0.$

28. $\dfrac{1}{a-b}+\dfrac{a-b}{x}=\dfrac{1}{a+b}+\dfrac{a+b}{x}.$

29. $\dfrac{1}{\dfrac{3(m+n)^2}{p^2x}-\dfrac{m+n}{p}}=\dfrac{p}{2(m+n)}.$

30. $\dfrac{y^2+2y-2}{y^2+5y+6}+\dfrac{y}{y+3}=\dfrac{y}{y+2}.$

31. $\dfrac{5}{2x+3}+\dfrac{7}{3x-4}=\dfrac{8x^2-13x-64}{6x^2+x-12}.$

32. $\dfrac{3a^2}{x^3-a^3}-\dfrac{1}{x-a}+\dfrac{a}{x^2+ax+a^2}=c.$

33. $\dfrac{1-2x}{3-4x}-\dfrac{5-6x}{7-8x}=\dfrac{8}{3}\cdot\dfrac{1-3x^2}{21-52x+32x^2}.$

34. $\dfrac{m-q}{x-n}+\dfrac{n-p}{x-q}=\dfrac{m-q}{x-p}+\dfrac{n-p}{x-m}.$

35. $\dfrac{3}{x-3}-\dfrac{2}{x-2}+\dfrac{8}{4x^2-20x+24}=0.$

36. $\dfrac{27}{x^3+27}-\dfrac{3}{x^2-3x+9}+\dfrac{1}{x+3}=0.$

37. $\dfrac{x-9}{x-5}-\dfrac{x-7}{x-2}-\dfrac{x-9}{x-4}=\dfrac{x-8}{x-5}-\dfrac{x-7}{x-4}-\dfrac{x-8}{x-2}.$

38. $3=\dfrac{(x+b-c)(x-b+c)}{(b+c+x)(b+c-x)}.$

PROBLEMS

1. Find a number such that if it is added to each term of the fraction $\frac{3}{8}$ and subtracted from each term of the fraction $\frac{13}{24}$ the results will be equal.

2. Make and solve a general problem of which 1 is a special case.

3. Three times one of two numbers is 4 times the other. If the sum of their squares is divided by the sum of the numbers, the quotient is $42\frac{6}{7}$ less than that obtained by dividing the sum of the squares by the difference of the numbers. Find the numbers.

4. The sum of two numbers less 2, divided by their difference, is 4, and the sum of their cubes divided by the difference of their squares is $1\frac{4}{5}$ times their sum. Find the numbers.

5. The circumference of the rear wheel of a carriage is 4 feet greater than that of the front wheel. In running one mile the front wheel makes 110 revolutions more than the rear wheel. Find the circumference of each wheel.

6. State and solve a general problem of which 5 is a special case, using b feet instead of one mile, letting the other numbers remain as they are in problem 5.

7. In going one mile the front wheel of a carriage makes 88 revolutions more than the rear wheel. If one foot is added to the circumference of the rear wheel, and 3 feet to that of the front wheel, the latter will make 22 revolutions more than the former. Find the circumference of each wheel.

8. State and solve a general problem of which 7 is a special case, using a instead of 88, letting the other numbers remain as they are.

9. The circumference of the front wheel of a carriage is a feet, and that of the rear wheel b feet. In going a certain distance the front wheel makes n revolutions more than the rear wheel. Find the distance.

10. State and solve a problem which is a special case of problem 9, using the formula just obtained.

11. There is a number consisting of two digits whose sum, divided by their difference, is 4. The number divided by the sum of its digits is equal to twice the digit in units' place plus $\frac{1}{8}$ of the digit in tens' place. Find the number.

12. There is a fraction such that if 3 is added to each of its terms, the result is $\frac{4}{5}$, and if 3 is subtracted from each of its terms, the result is $\frac{1}{2}$. Find the fraction.

13. State and solve a general problem of which 12 is a special case.

14. A and B working together can do a piece of work in 6 days. A can do it alone in 5 days less than B. How long will it require each when working alone?

15. State and solve a general problem of which 14 is a special case.

16. On her second westward trip the *Mauritania* traveled 625 knots in a certain time. If her speed had been 5 knots less per hour, it would have required $6\frac{1}{4}$ hours longer to cover the same distance. Find her speed per hour.

17. By increasing the speed a miles per hour, it requires b hours less to go c miles. Find the original speed. Show how problem 16 may be solved by means of the formula thus obtained.

18. A train is to run d miles in a hours. After going c miles a dispatch is received requiring the train to reach its destination b hours earlier. What must be the speed of the train for the remainder of the journey?

19. A man can row a miles down stream and return in b hours. If his rate up stream is c miles per hour less than down stream, find the rate of the current, and the rate of the boat in still water.

20. State and solve a special case of problem 19.

21. A can do a piece of work in a days, B can do it in b days, and C in c days. How long will it require all working together to do it?

22. Three partners, A, B, and C, are to divide a profit of p dollars. A had put in a dollars for m months, B had put in b dollars for n months, and C c dollars for t months. What share of the profit does each get?

23. State and solve a problem which is a special case of the preceding problem.

CHAPTER IX

RATIO, VARIATION, AND PROPORTION

RATIO AND VARIATION

166. In many important applications fractions are called **ratios.**

E.g. $\frac{3}{5}$ is called the ratio of 3 to 5 and is sometimes written 3 : 5.

It is to be understood that a ratio is the *quotient of two numbers* and hence is itself a *number.* We sometimes speak of the ratio of two magnitudes of the same kind, meaning thereby that these magnitudes are expressed in terms of a common unit and a ratio formed from the resulting *numbers.*

E.g. If, on measuring, the heights of two trees are found to be 25 feet and 35 feet respectively, we say the *ratio of their heights* is $\frac{25}{35}$ or $\frac{5}{7}$.

167. Two magnitudes are said to be **incommensurable** if there is no common unit of measure which is contained exactly an integral number of times in each.

E.g. If a and d are the lengths of the side and the diagonal of a square, then $d^2 = a^2 + a^2$, § 151, E. C. Hence, $\frac{a^2}{d^2} = \frac{1}{2}$ or $\frac{a}{d} = \frac{1}{\sqrt{2}}$. But since $\sqrt{2}$ is neither an *integer* nor a *fraction* (§ 108), it follows that a and d have no common measure, that is, they are *incommensurable.*

168. In many problems, especially in Physics, magnitudes are considered which are constantly changing. Number expressions representing such magnitudes are called **variables,** while those which represent fixed magnitudes are **constants.**

E.g. Suppose a body is moving at a uniform rate of 5 ft. per second. If t is the *number* of seconds from the time of starting and s the *number* of feet passed over, then s and t are *variables.*

The variables s and t, in case of uniform motion, have a *fixed ratio;* namely, in this example, $s:t=5$ for every pair of corresponding values of s and t throughout the period of motion.

169. When two variables are so related that for all pairs of corresponding values, their *ratio remains constant,* then each one is said to **vary directly as the other.**

E.g. If $s:t=k$ (a constant) then s varies directly as t, and t varies directly as s.

Variation is sometimes indicated by the symbol $\propto$. Thus $s \propto t$ means *s varies as t,* i.e. $\frac{s}{t}=k$ or $s=kt$.

170. When two variables are so related that for all pairs of corresponding values their *product remains constant,* than each one is said to **vary inversely as the other.**

E.g. Consider a rectangle whose area is A and whose base and altitude are b and h respectively. Then, $A=h\cdot b$.

If now the base is multiplied by 2, 3, 4, etc., while the altitude is divided by 2, 3, 4, etc., then the area will remain constant. Hence, b and h may both *vary* while A remains *constant.*

The relation $b\cdot h=A$ may be written $b=A\cdot\frac{1}{h}$ or $h=A\cdot\frac{1}{b}$. It may also be written $b:\frac{1}{h}=A$ or $h:\frac{1}{b}=A$, so that the ratio of either b or h to the *reciprocal* of the other is the constant A. For this reason one is said to *vary inversely as the other.*

171. If $y=kx^2$, k being constant and x and y variables, then y varies **directly as x^2.** If $y=\frac{k}{x^2}$, then y varies **inversely as x^2.** If $y=k\cdot wx$, then y **varies jointly as w and x.** If $y\propto wx$, then $y \propto w$ if x is constant and $y\propto x$ if w is constant. If $y=k\cdot\frac{w}{x}$, then **y varies directly as w and inversely as x.**

Example. The resistance offered by a wire to an electric current varies directly as its length and inversely as the area of its cross section.

If a wire $\frac{1}{8}$ in. in diameter has a resistance of r units per mile, find the resistance of a wire $\frac{1}{4}$ in. in diameter and 3 miles long.

Solution. Let R represent the resistance of a wire of length l and cross-section area $s = \pi \cdot (\text{radius})^2$. Then $R = k \cdot \frac{l}{s}$ where k is some constant. Since $R = r$ when $l = 1$ and $s = \pi(\frac{1}{16})^2$, we have

$$r = k \cdot \frac{1}{\frac{\pi}{256}} \text{ or } k = \frac{\pi r}{256}.$$

Hence, when $l = 3$ and $s = \pi(\frac{1}{8})^2$, we have,

$$R = \frac{\pi r}{256} \cdot \frac{3}{\frac{\pi}{64}} = \frac{3}{4} r.$$

That is, the resistance of three miles of the second wire is $\frac{3}{4}$ the resistance *per mile* of the first wire.

PROBLEMS

1. If $z \propto w$, and if $z = 27$ when $w = 3$, find the value of z when $w = 4\frac{1}{3}$.

2. If z varies jointly as w and x, and if $z = 24$ when $w = 2$ and $x = 3$, find z when $w = 3\frac{1}{2}$ and $x = 7$.

3. If z varies inversely as w, and if $z = 11$ when $w = 3$, find z when $w = 66$.

4. If z varies directly as w and inversely as x, and if $z = 28$ when $w = 14$ and $x = 2$, find z when $w = 42$ and $x = 3$.

5. If z varies inversely as the square of w, and if $z = 3$ when $w = 2$, find z when $w = 6$.

6. If q varies directly as m and inversely as the square of d, and $q = 30$ when $m = 1$ and $d = \frac{6}{100}$, find q when $m = 3$ and $d = 600$.

7. If $y^2 \propto x^3$, and if $y = 16$ when $x = 4$, find y when $x = 9$.

8. The weight of a triangle cut from a steel plate of uniform thickness varies jointly as its base and altitude. Find the base when the altitude is 4 and the weight 72, if it is known that the weight is 60 when the altitude is 5 and base 6.

9. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 13 ft., if a piece of radius 7 feet weighs 196 pounds.

10. If a body starts falling from rest, its velocity varies directly as the number of seconds during which it has fallen. If the velocity at the end of 3 seconds is 96.6 feet per second, find its velocity at the end of 7 seconds; of ten seconds.

11. If a body starts falling from rest, the total distance fallen varies directly as the square of the time during which it has fallen. If in 2 seconds it falls 64.4 feet, how far will it fall in 5 seconds? In 9 seconds?

12. The number of vibrations per second of a pendulum varies inversely as the square root of the length. If a pendulum 39.1 inches long vibrates once in each second, how long is a pendulum which vibrates 3 times in each second?

13. Illuminating gas in cities is forced through the pipes by subjecting it to pressure in the storage tanks. It is found that the volume of gas varies inversely as the pressure. A certain body of gas occupies 49,000 cu. ft. when under a pressure of 2 pounds per square inch. What space would it occupy under a pressure of $2\frac{4}{5}$ pounds per square inch?

14. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 15 feet from the stove moves up to 5 feet from it. How much will this increase the amount of heat received?

15. The weights of bodies of the same shape and of the same material vary as the cubes of corresponding dimensions. If a ball $3\frac{1}{4}$ inches in diameter weighs 14 oz., how much will a ball of the same material weigh whose diameter is $3\frac{1}{2}$ inches?

16. On the principle of problem 15, if a man 5 feet 9 inches tall weighs 165 pounds, what should be the weight of a man of similar build 6 feet tall?

PROPORTION

172. Definitions. The four numbers a, b, c, d are said to be **proportional** or to **form a proportion** if the ratio of a to b is equal to the ratio of c to d. That is, if $\frac{a}{b}=\frac{c}{d}$. This is also sometimes written $a:b::c:d$, and is read a is to b as c is to d.

The four numbers are called the **terms** of the proportion; the first and fourth are the **extremes**; the second and third the **means** of the proportion. The first and third are the **antecedents** of the ratios, the second and fourth the **consequents.**

If a, b, c, x are proportional, x is called the **fourth proportional** to a, b, c. If a, x, x, b are proportional, x is called a **mean proportional** to a and b, and b a **third proportional** to a and x.

173. If four numbers are proportional when taken in a *given order*, there are other orders in which they are also proportional.

E.g. If a, b, c, d are proportional in this order, they are also proportional in the following orders: a, c, b, d; b, a, d, c; b, d, a, c; c, a, d, b; c, d, a, b; d, c, b, a; and d, b, c, a.

Ex. 1. Write in the form of an equation the proportion corresponding to each set of four numbers given above, and show how each may be derived from $\frac{a}{b}=\frac{c}{d}$. See § 196, E. C.

Show first how to derive $\frac{a}{c}=\frac{b}{d}$ (1), and then $\frac{b}{a}=\frac{d}{c}$ (2).

Derive also $\frac{a+b}{a}=\frac{c+d}{c}$ (3), and $\frac{a-b}{a}=\frac{c-d}{c}$ (4).

In (1) the original proportion is said to be taken by **alternation,** and in (2) by **inversion**; in (3) by **composition,** and in (4) by **division.**

Ex. 2. From $\frac{a}{b}=\frac{c}{d}$ and (1), (2), (3), (4) obtain the following. See pp. 279–281, E. C.

$$\frac{a \pm b}{b}=\frac{c \pm d}{d}, \qquad \frac{a \pm b}{c \pm d}=\frac{a}{c}, \qquad \frac{a \pm c}{b \pm d}=\frac{a}{b}, \qquad \frac{a \pm b}{c+d}=\frac{a-b}{c \mp d}.$$

When the double sign occurs, the *upper* signs are to be read together and the *lower* signs together.

EXERCISES

1. What principles in the transformation of equations are involved (*a*) in taking a proportion by inversion, (*b*) by alternation, (*c*) by composition, (*d*) by division, (*e*) by composition and division?

2. From $\frac{a}{b}=\frac{c}{d}$ derive $\frac{c+d}{a+b}=\frac{d}{b}$ and state all the principles involved.

3. From $a:b::c:d$ show that $a^2:b^2::c^2:d^2$.

4. From $a:b::c:d$ and $m:n::r:s$ show that $am:bn::cr:ds$.

5. From $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}$ show that

$$\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n}=\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}.$$

Suggestion. Let $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}=k$.

6. If $\frac{a}{b}=\frac{c}{d}$ show that $\frac{ma-b}{mc-d}=\frac{a}{c}$.

7. If $\frac{x}{y}=\frac{z}{w}$ show that $\frac{x+ky}{x-ky}=\frac{z+kw}{z-kw}$.

8. If $\frac{a}{b}=\frac{c}{d}$ show that $\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc-nd}$.

9. (*a*) Find a fourth proportional to 17, 19, and 187.
   (*b*) Find a mean proportional between 6 and 54.
   (*c*) Find a third proportional to 27 and 189.

10. Find the unknown term in each of the following proportions:

$x:42::27:126$; $78:x::13:3$; $99:117::x:39$; $171:27::57:x$.

11. If s_1 and s_2 are two distances passed over by a body falling from rest in the time intervals t_1 and t_2, then $\frac{s_1}{s_2}=\frac{t_1^2}{t_2^2}$ (see problem 11, p. 138). If it is known that a body falls 144.9 feet in 3 seconds, how far will it fall in 8 seconds?

12. When a weight is attached to a spring balance the index is displaced a distance proportional to the weight. Thus, if d_1 and d_2 are displacements and w_1, w_2 the corresponding weights, then $\frac{d_1}{d_2}=\frac{w_1}{w_2}$. If a 2-pound weight displaces the index $\frac{1}{4}$ inch, how much will a 50-pound weight displace it?

13. The intensity of light is *inversely* proportional to the square of the distance from the source of the light. That is, if i_1 and i_2 are the measures of intensities at the distances d_1 and d_2, then $\frac{i_1}{i_2}=\frac{d_2^2}{d_1^2}$. If the intensity of a given light at a distance of 2 feet is 20 candle power, find the intensity at 5 feet?

14. If w_1 and w_2 are weights resting on the two ends of a beam, and if the distances from the fulcrum are d_1 and d_2 respectively, then the beam will balance when $\frac{w_1}{w_2}=\frac{d_2}{d_1}$. That is, the weights are *inversely* proportional to the distances.

If a stone weighing 850 pounds at a distance of 1 foot from the fulcrum is to be balanced by a 50-pound weight, where should the weight be applied?

15. Find where the fulcrum should be in order that two boys weighing 110 and 80 pounds respectively may balance on the ends of a 16-foot plank.

16. The weight of a body above the earth's surface varies inversely as the square of its distance from the earth's center. If an object weighs 2000 pounds at the earth's surface, what would be its weight if it were 12,000 miles above the center of the earth, the radius of the earth being 4000 miles?

CHAPTER X

EXPONENTS AND RADICALS

FRACTIONAL AND NEGATIVE EXPONENTS

174. The meaning heretofore attached to the word *exponent* cannot apply to a fractional or negative number.

E.g. Such an exponent as $\frac{2}{3}$ or -5 cannot indicate the *number of times* a base is used as a factor.

It is possible, however, to interpret fractional and negative exponents in such a way *that the laws of operations which govern positive integral exponents shall apply to these also.*

175. The laws for positive integral exponents are:

I. $a^m \cdot a^n = a^{m+n}$. § 43

II. $a^m \div a^n = a^{m-n}$. § 46

III. $(a^m)^n = a^{mn}$. § 115

IV. $(a^m \cdot b^n)^p = a^{mp}b^{np}$. § 116

V. $(a^m \div s^n)^p = a^{mp} \div s^{np}$. § 117

176. Assuming Law I to hold for positive fractional exponents and letting r and s be positive integers, we determine as follows the meaning of $b^{\frac{r}{s}}$ (read *b exponent r divided by s*).

By definition, $\left(b^{\frac{r}{s}}\right)^s = b^{\frac{r}{s}} \cdot b^{\frac{r}{s}} \cdots$ to s factors,

which by Law I $\quad = b^{\frac{r}{s}+\frac{r}{s}+\cdots \text{ to } s \text{ terms}} = b^{\frac{r}{s}\cdot s} = b^r$.

Hence, $b^{\frac{r}{s}}$ is one of the s equal factors of b^r.

That is, $b^{\frac{r}{s}} = \sqrt[s]{b^r}$, and in particular $b^{\frac{1}{s}} = \sqrt[s]{b}$. See § 114

Similarly, from $\left(b^{\frac{1}{s}}\right)^r = b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdots$ to r factors, $= b^{\frac{r}{s}}$,

we show that $$b^{\frac{r}{s}} = \left(b^{\frac{1}{s}}\right)^r = (\sqrt[s]{b})^r.$$

Hence, $$b^{\frac{r}{s}} = \sqrt[s]{b^r} = (\sqrt[s]{b})^r.$$ See § 119

Thus a positive fractional exponent means *a root of a power or a power of a root, the numerator indicating the power and the denominator indicating the root.*

E.g. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$; $8^{\frac{2}{3}} = \sqrt[3]{64} = 4$, or $(\sqrt[3]{8})^2 = 2^2 = 4$.

177. Assuming Law I to hold also for negative exponents, and letting t be a positive number, integral or fractional, we determine as follows the meaning of b^{-t} (*read b exponent negative t*).

By Law I, $$b^t \cdot b^{-t} = b^0 = 1.$$ § 46

Therefore, $$b^{-t} = \frac{1}{b^t}.$$ § 11

Hence a number with a negative exponent means *the same as the reciprocal of the number with a positive exponent of the same absolute value.*

E.g. $a^{-2} = \frac{1}{a^2}$; $\quad 4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{2^3} = \frac{1}{8}$.

178. It thus appears that fractional and negative exponents simply provide *new ways of indicating operations already well known.* Sometimes one notation is more convenient and sometimes the other.

Fractional and negative exponents are also called *powers.*

E.g. $x^{\frac{2}{3}}$ may be read *x to the* $\frac{2}{3}$ *power*, and x^{-4} may be read *x to the* − *4th power.*

The limitations as to principal roots and the sign of the base, imposed in theorems on powers and roots in Chapter VI, necessarily apply to the corresponding theorems in this chapter. See §§ 114–122.

In any algebraic expression, radical signs may now be replaced by fractional exponents, or fractional exponents by radical signs.

In a fraction, any *factor* may be changed from numerator to denominator, or from denominator to numerator, by changing the sign of its exponent.

Ex. 1. $\sqrt[3]{x^2} + 3\sqrt[5]{x^3}\cdot\sqrt{y} + 5\sqrt[7]{x^4}\sqrt[3]{y^2} \equiv x^{\frac{2}{3}} + 3x^{\frac{3}{5}}y^{\frac{1}{2}} + 5x^{\frac{4}{7}}y^{\frac{2}{3}}$.

Ex. 2. $\dfrac{ab}{x^2} = abx^{-2}$, since $abx^{-2} = ab\cdot\dfrac{1}{x^2} = \dfrac{ab}{x^2}$.

Ex. 3. $ab^{-3}c^2 = ac^2\cdot\dfrac{1}{b^3} = \dfrac{ac^2}{b^3}$.

Ex. 4. $32^{-\frac{4}{5}} = \dfrac{1}{32^{\frac{4}{5}}} = \dfrac{1}{(\sqrt[5]{32})^4} = \dfrac{1}{2^4} = \dfrac{1}{16}$.

EXERCISES

(*a*) In the expressions containing radicals on p. 454, replace these by fractional exponents.

(*b*) Replace all positive fractional exponents on this page by radicals.

(*c*) Change all expressions containing negative exponents to equivalent expressions having only positive exponents.

179. Fractional and negative exponents have been defined so as to conform to Law I, §§ 176, 177. We now show that when so defined they also conform to Laws II, III, IV, and V.

To verify Law II. Since by Law I, $a^{m-n}\cdot a^n = a^m$, for m and n integral or fractional, positive or negative, it follows by § 11 that $a^m \div a^n = a^{m-n}$ for all rational exponents.

To verify Law III. Let r and s be positive integers, and let k be any positive or negative integer or fraction. Then we have:

(1) $(a^k)^{\frac{r}{s}} = \sqrt[s]{(a^k)^r} = \sqrt[s]{a^{kr}} = a^{\frac{rk}{s}} = a^{\frac{r}{s}\cdot k}$, by §§ 176, 115.

(2) $(a^k)^{-\frac{r}{s}} = \dfrac{1}{(a^k)^{\frac{r}{s}}} = \dfrac{1}{a^{\frac{r}{s}\cdot k}} = a^{-\frac{r}{s}\cdot k}$, by § 177 and (1).

Hence $(a^k)^n = a^{nk}$ for all rational values of n and k.

To verify Law IV. Let m and n be positive or negative integers or fractions, and let r and s be positive integers, then we have

$$(1)\quad (a^m b^n)^{\frac{r}{s}} = \sqrt[s]{(a^m b^n)^r} = \sqrt[s]{a^{mr}b^{nr}}, \qquad \text{by §§ 176, 115,}$$

$$= \sqrt[s]{a^{mr}} \cdot \sqrt[s]{b^{nr}} = a^{\frac{r}{s}\cdot m} \cdot b^{\frac{r}{s}\cdot n}, \qquad \text{by §§ 120, 176.}$$

$$(2)\quad (a^m b^n)^{-\frac{r}{s}} = \frac{1}{(a^m b^n)^{\frac{r}{s}}} = \frac{1}{a^{\frac{r}{s}\cdot m} \cdot b^{\frac{r}{s}\cdot n}} = a^{-\frac{r}{s}\cdot m} \cdot b^{-\frac{r}{s}\cdot n}, \text{ by § 177 and (1).}$$

Hence $(a^m b^n)^p = a^{pm}b^{pn}$ for all rational values of m, n, and p.

To verify Law V. We have $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$ for all rational values of m, n, and p, since

$$\left(\frac{a^m}{b^n}\right)^p = (a^m \cdot b^{-n})^p = a^{mp} \cdot b^{-np} = \frac{a^{mp}}{b^{np}}, \text{ § 177 and Law IV.}$$

180. From Laws III, IV, and V, it follows that any monomial is affected with any exponent by multiplying the exponent of each factor of the monomial by the given exponent.

Ex. 1. $(a^{\frac12}b^{-2}c^3)^{-\frac23} = a^{-\frac23\cdot\frac12}b^{-\frac23\cdot -2}c^{-\frac23\cdot 3} = a^{-\frac13}b^{\frac43}c^{-2}.$

Ex. 2. $\left(\dfrac{3a^2x^6}{by^4}\right)^{-\frac12} = \dfrac{3^{-\frac12}a^{-1}x^{-3}}{b^{-\frac12}y^{-2}} = \dfrac{b^{\frac12}y^2}{3^{\frac12}ax^3}.$

Ex. 3. $\left(\dfrac{8x^9}{27y^6}\right)^{-\frac13} = \left(\dfrac{27y^6}{8x^9}\right)^{\frac13} = \dfrac{27^{\frac13}y^2}{8^{\frac13}x^3} = \dfrac{3y^2}{2x^3}.$

EXERCISES

Perform the operations indicated by the exponents in each of the following, writing the results without negative exponents and in as simple form as possible:

1. $(\frac{16}{25})^{-\frac12}$.	5. $(x^{-\frac23}y^{\frac43})^{-\frac34}$.	9. $(\frac{27}{8})^{-\frac23}$.	13. $(.0009)^{\frac32}$.
2. $(\frac{27}{64})^{-\frac13}$.	6. $25^{\frac32}$.	10. $(\frac{8}{27})^{\frac23}$.	14. $(.027)^{\frac13}$.
3. $(\frac{25}{36})^{\frac32}$.	7. $25^{-\frac32}$.	11. $(0.25)^{\frac12}$.	15. $(32a^{-5}b^{10})^{\frac15}$.
4. $(27a^{-9})^{\frac13}$.	8. 25^0.	12. $(0.25)^{-\frac12}$.	16. $8^{\frac13}\cdot 4^{-\frac12}$.

17. $\left(\dfrac{a^{-8}}{16}\right)^{-\frac{1}{4}}$.

18. $(27\,x^6y^{-3}z^{-1})^{-\frac{1}{3}}$.

19. $(\frac{1}{32})^{-\frac{1}{5}}\,(\frac{1}{81})^{-\frac{3}{4}}$.

20. $\sqrt[3]{\frac{512}{729}}\cdot(\frac{256}{729})^{-\frac{1}{2}}$.

21. $(-\frac{243}{32})^{\frac{1}{5}} \div (\frac{16}{81})^{-\frac{1}{4}}$.

22. $\left(\dfrac{x^3y^{-4}}{x^{-2}y}\right)^3\left(\dfrac{x^{-3}y^2}{xy^{-1}}\right)^5$.

23. $\sqrt[4]{81\,a^{-4}b^8}(-27\,a^3b^{-6})^{-\frac{1}{3}}$.

24. $\left(\dfrac{m^{-1}n}{m^{\frac{1}{2}}n^{-\frac{2}{3}}}\right)^{-2} \div \left(\dfrac{m^{-3}}{n^{-1}}\right)^{-\frac{1}{3}}$.

25. $\left(\dfrac{a^3b^{-2}}{a^{-2}b^3}\right)^{\frac{1}{5}} \div \left(\dfrac{a^3b^{-3}}{a^{-3}b^3}\right)^{-1}$.

26. $\sqrt{16\,a^{-4}b^{-6}}\cdot\sqrt[3]{8\,a^3b^{-6}}$.

27. $(-2^{-2}a^{-3}b^{-6})^{-\frac{1}{3}}(-2^{-\frac{1}{3}}a^{-\frac{1}{2}}b^{-1})^2$.

28. $\left(\dfrac{81\,r^{-16}s^4t}{625\,r^4s^8t}\right)^{\frac{1}{4}}\left(\dfrac{9\,r^2s^{-4}t}{25\,r^{10}s^0t}\right)^{-\frac{1}{2}}$.

29. Prove Law III in detail for the following cases:

$$(1)\ (a^{\frac{m}{n}})^{-\frac{r}{s}},\quad (2)\ (a^{-\frac{m}{n}})^{\frac{r}{s}},\quad (3)\ (a^{-\frac{m}{n}})^{-\frac{r}{s}}.$$

30. Prove Law IV in detail for the following cases:

$$(1)\ (a^{-k}b^{-l})^{\frac{r}{s}},\quad (2)\ (a^{-k}b^{-l})^{\frac{r}{s}}.$$

Multiply:

31. $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-1}-y^{-1}$.

32. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

33. $x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$ by $x^{\frac{1}{5}}-y^{\frac{1}{5}}$.

34. $\sqrt[4]{a^3}+\sqrt[5]{b^2}$ by $\sqrt[4]{a^3}-\sqrt[5]{b^2}$.

35. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

36. $x-3\,x^{\frac{2}{3}}y^{-\frac{1}{2}}+3\,x^{\frac{1}{3}}y^{-1}-y^{-\frac{3}{2}}$ by $x^{\frac{2}{3}}-2\,x^{\frac{1}{3}}y^{-\frac{1}{2}}+y^{-1}$.

37. $x^{\frac{3}{2}}+xy^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^{\frac{9}{4}}$ by $x^{\frac{1}{2}}-y^{\frac{3}{4}}$.

Divide:

38. $x^2-x^{\frac{11}{6}}y+x^{\frac{3}{2}}y-x^{\frac{1}{2}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{3}}y+y$.

39. $3\,a^{\frac{7}{4}}-ab^{\frac{2}{3}}+4\,ab^2-3\,a^{\frac{3}{4}}b+b^{\frac{5}{3}}-4\,b^3$ by $3\,a^{\frac{3}{4}}-b^{\frac{2}{3}}+4\,b^2$.

40. $x^2-3\,x^{\frac{5}{3}}+6\,x^{\frac{4}{3}}-7\,x+6\,x^{\frac{2}{3}}-3\,x^{\frac{1}{3}}+1$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$.

41. $4\,x^{\frac{5}{4}}b^{-2}-17\,x^{\frac{1}{4}}b^2+16\,x^{-\frac{3}{4}}b^6$ by $2\,x^{\frac{1}{2}}-b^2-4\,x^{-\frac{1}{2}}b^4$.

Find the square root of:

42. $4x^2 - 4xy^{\frac{1}{3}} + 4xz^{-\frac{1}{2}} + y^{\frac{2}{3}} - 2y^{\frac{1}{3}}z^{-\frac{1}{2}} + z^{-1}$.

43. $a^{-\frac{2}{3}} - 2a^{-\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}} + 2a^{-\frac{1}{3}}c^2 + c^4 - 2b^{\frac{2}{3}}c^2$.

44. $b^{-\frac{1}{2}} - 2b^{-\frac{1}{4}}c^{\frac{1}{3}} + c^{\frac{2}{3}} + 2b^{-\frac{1}{4}}d^{\frac{1}{5}} + 2b^{-\frac{1}{4}}e^{-\frac{1}{2}} - 2c^{\frac{1}{3}}d^{\frac{1}{5}} + d^{\frac{2}{5}}$
$+ 2d^{\frac{1}{5}}e^{-\frac{1}{2}} - 2c^{\frac{1}{3}}e^{-\frac{1}{2}} + e^{-1}$.

Find the cube root of:

45. $\frac{1}{8}a^3 - \frac{3}{2}a^2b^{\frac{1}{2}} + 6ab - 8b^{\frac{3}{2}}$.

46. $a^6 - 3a^5 + 5a^3 - 3a - 1$.

47. $a^{-1} + 3a^{-\frac{2}{3}}b^{\frac{2}{3}} + 3a^{-\frac{1}{3}}b^{\frac{4}{3}} + b^{\frac{6}{3}}$.

REDUCTION OF RADICAL EXPRESSIONS

181. An expression containing a root indicated by the radical sign or by a fractional exponent is called a **radical expression.** The expression whose root is indicated is the **radicand.**

E.g. $\sqrt[3]{5}$ and $(1 + x)^{\frac{2}{3}}$ are radical expressions. In each case the **index** of the radical is 3.

The reduction of a radical expression consists in *changing its form without changing its value.*

Each reduction is based upon one or more of the Laws I to V, § 175, as extended in § 179.

182. To remove a factor from the radicand. This reduction is possible only when the radicand contains a factor which is a perfect power of the degree indicated by the index of the root, as shown in the following examples:

Ex. 1. $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

Ex. 2. $(a^3x^2y^6)^{\frac{1}{3}} = (a^3y^6 \cdot x^2)^{\frac{1}{3}} = (a^3y^6)^{\frac{1}{3}} \cdot (x^2)^{\frac{1}{3}} = ay^2x^{\frac{2}{3}}$.

This reduction involves Law IV, and may be written in symbols thus:

$$\sqrt[r]{x^{kr}y} = \sqrt[r]{x^{kr}}\,\sqrt[r]{y} = x^k\,\sqrt[r]{y}.$$

EXERCISES

In the expressions on p. 454, remove factors from the radicands where possible.

In the case of negative fractional exponents, first reduce to equivalent expressions containing only positive exponents.

183. To introduce a factor into the radicand. This process simply retraces the steps of the foregoing reduction, and hence also involves Law IV.

Ex. 1. $6\sqrt{2} = \sqrt{6^2}\cdot\sqrt{2} = \sqrt{36\cdot 2} = \sqrt{72}.$ See § 112

Ex. 2. $ay^2x^{\frac{2}{3}} = \sqrt[3]{(ay^2)^3}\cdot\sqrt[3]{x^2} = \sqrt[3]{(ay^2)^3x^2} = \sqrt[3]{a^3y^6x^2}.$

Ex. 3. $x\sqrt[r]{y} = \sqrt[r]{x^r}\,\sqrt[r]{y} = \sqrt[r]{x^ry}.$

EXERCISES

In the expressions on p. 454, introduce into the radicand any factor which appears as a coefficient of a radical.

184. To reduce a fractional radicand to the integral form. This reduction involves Law IV or Law V, and may always be accomplished.

Ex. 1. $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \sqrt{\frac{1}{25}\cdot 15} = \frac{1}{5}\sqrt{15}.$ Law IV

Ex. 2. $\left(\dfrac{a-b}{a+b}\right)^{\frac{1}{2}} = \left(\dfrac{a^2-b^2}{(a+b)^2}\right)^{\frac{1}{2}} = \dfrac{(a^2-b^2)^{\frac{1}{2}}}{[(a+b)^2]^{\frac{1}{2}}} = \dfrac{(a^2-b^2)^{\frac{1}{2}}}{a+b}.$ Law V

Ex. 3. $\dfrac{\sqrt[3]{40}}{\sqrt[3]{5}} = \sqrt[3]{\dfrac{40}{5}} = \sqrt[3]{8} = 2.$

In symbols, we have $\sqrt[r]{\dfrac{a}{b}} = \sqrt[r]{\dfrac{ab^{r-1}}{b^r}} = \dfrac{\sqrt[r]{ab^{r-1}}}{\sqrt[r]{b^r}} = \dfrac{1}{b}\sqrt[r]{ab^{r-1}}.$

EXERCISES

In the expressions on p. 454, reduce each fractional radicand to the integral form.

In case negative exponents are involved, first reduce to equivalent expressions containing only positive exponents.

185. To reduce a radical to an equivalent radical of lower index. This reduction is effective when the radicand is a perfect power corresponding to *some factor of the index.*

Ex. 1. $\sqrt[6]{8}=8^{\frac{1}{6}}=8^{(\frac{1}{3\cdot 2})}=(8^{\frac{1}{3}})^{\frac{1}{2}}=2^{\frac{1}{2}}=\sqrt{2}.$

Ex. 2. $\sqrt[4]{a^2+2\,ab+b^2}=\sqrt{\sqrt{a^2+2\,ab+b^2}}=\sqrt{a+b}.$

This reduction involves Law III as follows:

$$(x^{\frac{1}{r}})^{\frac{1}{s}}=(x^{\frac{1}{s}})^{\frac{1}{r}}=x^{\frac{1}{rs}},$$

from which we have

$$\sqrt[rs]{x}=\sqrt[r]{\sqrt[s]{x}}=\sqrt[s]{\sqrt[r]{x}}. \qquad \text{See § 114}$$

By this reduction a root whose index is a composite number is made to depend upon roots of lower degree.

E.g. A fourth root may be found by taking the square root twice; a sixth root, by taking a square root and then a cube root, etc. In the case of *literal* expressions this can be done only when the radicand is a perfect power of the degree indicated by the index of the root.

But when the radicand is expressed in Arabic figures, such roots may in any case be approximated as in § 127.

EXERCISES

In the expressions on p. 454, make the reduction above indicated where possible.

In the case of arithmetic radicands, approximate to two places of decimals such roots as can be made to depend upon square and cube roots.

186. To reduce a radical to an equivalent radical of higher index. This reduction is possible whenever the required index is a *multiple* of the given index. It is based on Law III as follows:

$$x^{\frac{r}{s}}=(x^{\frac{r}{s}})^{\frac{t}{t}}=x^{\frac{rt}{st}}. \qquad \text{See § 179}$$

Ex. 1. $\sqrt{a}=a^{\frac{1}{2}}=(a^{\frac{1}{2}})^{\frac{3}{3}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}.$

Ex. 2. $\sqrt[3]{b}=b^{\frac{1}{3}}=b^{\frac{2}{6}}=\sqrt[6]{b^2}.$

Definition. Two radical expressions are said to be of the **same order** when their indicated roots have the *same index.*

By the above reduction two radicals of *different* orders may be changed to equivalent radicals of the *same* order, namely, a common multiple of the given indices.

E.g. $\sqrt{a}$ and $\sqrt[3]{b}$ in Exs. 1 and 2 above.

EXERCISES

In Exs. 3, 4, 6, 17, 18, 23, 28, 30, on p. 454, reduce the corresponding expressions in the first and second columns to equivalent radicals of the same order.

187. In general, radical expressions should be at once reduced so that the *order is as low* as possible and the *radicand is integral and as small* as possible. A radical is then said to be in its **simplest form.**

ADDITION AND SUBTRACTION OF RADICALS

188. Definition. Two or more radical expressions are said to be **similar** when they are of the same order and have the same radicands.

E.g. $3\sqrt{7}$ and $5\sqrt{7}$ are similar radicals as are also $a\sqrt[7]{x^4}$ and $bx^{\frac{4}{7}}$.

If two radicals can be reduced to similar radicals, they may be added or subtracted according to § 10.

Ex. 1. Find the sum of $\sqrt{8}$, $\sqrt{50}$, and $\sqrt{98}$.

By § 182, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{98} = 7\sqrt{2}$.

Hence $\sqrt{8} + \sqrt{50} + \sqrt{98} = 2\sqrt{2} + 5\sqrt{2} + 7\sqrt{2} = 14\sqrt{2}$.

Ex. 2. Simplify $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}}$.

By § 184, $\sqrt{\frac{1}{5}} = \frac{1}{5}\sqrt{5}$, $\sqrt{20} = 2\sqrt{5}$, $\sqrt{3\frac{1}{5}} = \sqrt{\frac{16}{5}} = 4\sqrt{\frac{1}{5}} = \frac{4}{5}\sqrt{5}$.

Hence $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}} = \frac{1}{5}\sqrt{5} - 2\sqrt{5} + \frac{4}{5}\sqrt{5} = -\sqrt{5}$.

If two radicals cannot be reduced to equivalent similar radicals, their sum can only be indicated.

E.g. The sum of $\sqrt{2}$ and $\sqrt[3]{5}$ is $\sqrt{2} + \sqrt[3]{5}$.

Observe, however, that

$$\sqrt{10} + \sqrt{6} = \sqrt{2} \cdot \sqrt{5} + \sqrt{2} \cdot \sqrt{3} = \sqrt{2}\,(\sqrt{5} + \sqrt{3}).$$

EXERCISES

(*a*) In Exs. 1, 2, 5, 7, 8, 19, 20, 21, p. 454, reduce each pair so as to involve similar radicals and add them.

(*b*) Perform the following indicated operations:

1. $\sqrt{28} + 3\sqrt{7} - 2\sqrt{63}$.
2. $\sqrt[3]{24} - \sqrt[3]{81} - \sqrt[3]{\frac{3}{125}}$.
3. $\sqrt[5]{a^6} + \sqrt[5]{a^{11}} - \sqrt[5]{32\,a}$.
4. $2\sqrt{48} - 3\sqrt{12} + 3\sqrt{\frac{1}{3}}$.
5. $\sqrt{\frac{1}{7}} + \sqrt{63} + 5\sqrt{7}$.
6. $\sqrt{99} - 11\sqrt{\frac{1}{11}} + \sqrt{44}$.
7. $2\sqrt{\frac{4}{7}} + 3\sqrt{\frac{9}{7}} + \sqrt{175}$.
8. $\sqrt[4]{\frac{9}{625}} + 6\sqrt{\frac{1}{3}} - \sqrt{12}$.
9. $\sqrt[6]{9} + \sqrt[9]{27} + \sqrt[3]{-24}$.
10. $x^2 + \sqrt{a^3 + a^2b} - \sqrt{(a^2 - b^2)(a - b)}$.

MULTIPLICATION OF RADICALS

189. Radicals of the *same order* are multiplied according to § 120 by multiplying the radicands. If they are not of the same order, they may be reduced to the same order according to § 186.

E.g. $\sqrt{a} \cdot \sqrt[3]{b} = a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = \sqrt[6]{a^3} \cdot \sqrt[6]{b^2} = \sqrt[6]{a^3b^2}$.

In many cases this reduction is not desirable. Thus, $\sqrt[3]{x^2} \cdot \sqrt{y^3}$ is written $x^{\frac{2}{3}}y^{\frac{3}{2}}$ rather than $\sqrt[6]{x^4y^9}$.

Radicals are multiplied by adding exponents when they are reduced to the *same base* with fractional exponents, § 176.

E.g. $\sqrt[5]{x^2} \cdot \sqrt{x^3} = x^{\frac{2}{5}} \cdot x^{\frac{3}{2}} = x^{\frac{2}{5}+\frac{3}{2}} = x^{\frac{19}{10}}$.

190. The principles just enumerated are used in connection with § 10 in multiplying polynomials containing radicals.

Ex. 1.

$$\begin{array}{lll} 3\sqrt{2} & + 2\sqrt{5} & \\ 2\sqrt{2} & - 3\sqrt{5} & \\ \hline 6\cdot 2 & + 4\sqrt{10} & \\ & - 9\sqrt{10} & - 6\cdot 5 \\ \hline 12 & - 5\sqrt{10} & - 30 \end{array}$$

Ex. 2.

$$\begin{array}{lll} 3\sqrt{2} & + 2\sqrt{5} & \\ 3\sqrt{2} & - 2\sqrt{5} & \\ \hline 9\cdot 2 & + 6\sqrt{10} & \\ & - 6\sqrt{10} & - 4\cdot 5 \\ \hline 18 & & - 20 \end{array}$$

Hence $(2\sqrt{2} + 2\sqrt{5})(2\sqrt{2} - 3\sqrt{5}) = -18 - 5\sqrt{10}$,

and $(3\sqrt{2} + 2\sqrt{5})(3\sqrt{2} - 2\sqrt{5}) = 18 - 20 = -2$.

EXERCISES

(*a*) In Exs. 21 to 38, p. 454, find the products of the corresponding expressions in the two columns.

(*b*) Find the following products:

1. $(3 + \sqrt{11})(3 - \sqrt{11})$.
2. $(3\sqrt{2} + 4\sqrt{5})(4\sqrt{2} - 5\sqrt{5})$.
3. $(2 + \sqrt{3} + \sqrt{5})(3 + \sqrt{3} - \sqrt{5})$.
4. $(3\sqrt{2} - 2\sqrt{18} + 2\sqrt{7})(2\sqrt{2} - \sqrt{18} - \sqrt{7})$.
5. $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})(a^2 + ab + b^2)$.
6. $(\sqrt{\sqrt{13} + 3})(\sqrt{\sqrt{13} - 3})$.
7. $(\sqrt{2 + 3\sqrt{5}})(\sqrt{2 + 3\sqrt{5}})$.
8. $(3a - 2\sqrt{x})(4a + 3\sqrt{x})$.
9. $(3\sqrt{3} + 2\sqrt{6} - 4\sqrt{8})(3\sqrt{3} - 2\sqrt{6} + 4\sqrt{8})$.
10. $(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$.

11. $(a-\sqrt{b}-\sqrt{c})(a+\sqrt{b}+\sqrt{c})$.

12. $(2\sqrt{\frac{2}{3}}+3\sqrt{\frac{3}{6}}+4\sqrt{\frac{3}{2}})(2\sqrt{\frac{2}{3}}-5\sqrt{\frac{3}{6}})$.

13. $(\sqrt[3]{a^2}+\sqrt[3]{b^2})(\sqrt[6]{a^2}+\sqrt[3]{a^2b^2}+\sqrt[9]{b^3})$.

14. $(\sqrt[4]{x^3}-y^3)^3$.

DIVISION OF RADICALS

191. Radicals are divided in accordance with Laws II and V. That is, the exponents are *subtracted* when the *bases* are the *same*, and the bases are *divided* when the *exponents* are the *same*. See §§ 179, 121.

Ex. 1. $\sqrt[5]{x^2}\div\sqrt{x^3}=x^{\frac{2}{5}}\div x^{\frac{3}{2}}=x^{\frac{2}{5}-\frac{3}{2}}=x^{-\frac{11}{10}}$.

Ex. 2. $x^{\frac{2}{3}}\div y^{\frac{2}{3}}=\left(\frac{x}{y}\right)^{\frac{2}{3}}=(xy^{-1})^{\frac{2}{3}}=\sqrt[3]{x^2y^{-2}}$.

Ex. 3. $\sqrt{a}\div\sqrt[3]{b}=a^{\frac{3}{6}}\div b^{\frac{2}{6}}=\left(\frac{a^3}{b^2}\right)^{\frac{1}{6}}=\sqrt[6]{a^3b^{-2}}$.

EXERCISES

(*a*) In each of the Exs. 1 to 20, on p. 454, divide the expression in the first column by that in the second.

(*b*) Perform the following divisions:

1. $(\sqrt{a^3}+2\sqrt{a^5}-3\sqrt{a})\div 6\sqrt{a}$.

2. $(\sqrt{a}+\sqrt[3]{b}-c)\div\sqrt{c}$.

3. $(2\sqrt[3]{9}+3\sqrt[3]{12}-4\sqrt[3]{15})\div\sqrt[3]{3}$.

4. $(4\sqrt[5]{7}-8\sqrt[5]{21}+6\sqrt[5]{42})\div 2\sqrt[5]{7}$.

EXERCISES

1. $3\sqrt{45}$, $2\sqrt{125}$.
2. $a^{\frac{2}{3}}$, $a^{\frac{5}{3}}$.
3. $x^{\frac{2}{5}}$, $x^{\frac{5}{3}}$.
4. $3\sqrt{x^2y}$, $2\sqrt[3]{x^4}$.
5. $d\sqrt[3]{a^2b^5}$, $c\sqrt[3]{a^5b^2}$.
6. $7\sqrt{(a+b)^3}$, $11\sqrt[3]{(a-b)^6}$.
7. $\sqrt[3]{a^4}$, $a^{\frac{7}{3}}$.
8. $n\sqrt{m^5}$, $m^{\frac{3}{2}}$.
9. $\sqrt{\frac{7}{2}}$, $\sqrt{\frac{8}{7}}$.
10. $\sqrt{\frac{11}{72}}$, $\sqrt{\frac{23}{54}}$.
11. $\sqrt{\frac{36}{27}}$, $\sqrt{\frac{13}{18}}$.
12. $\dfrac{1}{b^{-\frac{1}{2}}}$, $r\sqrt{b^3}$.
13. $a^{-2}b^3$, $\dfrac{a^3}{b^4}$.
14. $\dfrac{3}{m^{-7}}$, $\dfrac{n^{-\frac{1}{3}}m^{-1}}{4}$.
15. $\dfrac{3a^{-\frac{3}{2}}}{c^{-2}b^3}$, $\dfrac{2a^4b^{-2}}{c^3}$.
16. $\sqrt[3]{\frac{3}{4}}$, $\sqrt[3]{\frac{1}{4}}$.
17. $\sqrt[4]{\frac{3}{8}}$, $\sqrt[6]{\frac{2}{9}}$.
18. $\sqrt[9]{\frac{3}{64}}$, $\sqrt[6]{\frac{2}{7}}$.
19. $18^{\frac{1}{2}}$, $\sqrt{32}$.
20. $\sqrt{12}$, $48^{\frac{1}{2}}$.
21. $3(50)^{\frac{1}{2}}$, $4(72)^{\frac{1}{2}}$.
22. $a^{\frac{3}{4}}$, $a^{\frac{5}{3}}$.
23. $ax^{\frac{7}{2}}$, $bx^{\frac{3}{4}}$.
24. $a^{\frac{2}{3}}b^{\frac{7}{5}}$, $a^{\frac{3}{2}}b^{\frac{5}{7}}$.
25. $m^{\frac{3}{4}}n^{\frac{4}{5}}l^{\frac{7}{8}}$, $m^{\frac{3}{2}}n^{\frac{2}{3}}l^{\frac{4}{5}}$.
26. $5\sqrt[5]{a^6b^4c^9}$, $3\sqrt[5]{a^3b^7c^{13}}$.
27. $\sqrt[2]{a^3}\sqrt[4]{b^{10}}$, $\sqrt[3]{b^9}\sqrt[4]{a^6}$.
28. $8^{\frac{2}{3}}$, $16^{\frac{3}{2}}$.
29. $25^{-\frac{1}{2}}$, $125^{-\frac{4}{3}}$.
30. $9^{\frac{3}{2}}$, $8^{\frac{2}{3}}$.
31. $(\frac{1}{9})^{\frac{1}{3}}$, $(\frac{1}{27})^{-\frac{1}{3}}$.
32. $\dfrac{5x^3y^5}{m^{-3}n^{-5}}$, $\dfrac{x^3y^5z^2}{m^{-1}n^{-2}l^{-3}}$.
33. $\dfrac{3a^{-\frac{3}{2}}b^{\frac{3}{2}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$, $\dfrac{4a^{-\frac{5}{3}}b^{-\frac{3}{2}}}{5a^{-\frac{1}{2}}b^{-\frac{1}{3}}}$.
34. $\dfrac{c^{-\frac{1}{2}}d^{-\frac{1}{3}}}{d^{-\frac{1}{4}}c^{-\frac{1}{5}}}$, $\dfrac{d^{\frac{3}{4}}c^0}{d^{-\frac{1}{4}}c^{-1}}$.
35. $5(a+b)^{-\frac{3}{2}}$, $3(a+b)^{-\frac{2}{3}}$.
36. $(-64)^{\frac{5}{6}}$, $-64^{\frac{5}{6}}$.
37. $(\frac{1}{27})^{\frac{1}{3}}$, $(\frac{1}{16})^{-\frac{3}{4}}$.
38. $\dfrac{1}{b}\sqrt[3]{\dfrac{a^3}{b}}$, $\frac{1}{3}\sqrt[3]{\frac{27}{5}}$.
39. $\sqrt[3]{4a^3-12a^2b+12ab^2-4b^3}$.
40. $\sqrt{(3a-2b)(9a^2-4b^2)}$.

192. Rationalizing the divisor. In case division by a radical expression cannot be carried out as in the foregoing examples, it is desirable to *rationalize* the denominator when possible.

Ex. 1. $\dfrac{\sqrt{2}}{\sqrt{5}} = \dfrac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \dfrac{\sqrt{10}}{5}.$

Ex. 2. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})} = \dfrac{a+\sqrt{ab}}{a-b}.$

Evidently this is always possible when the divisor is a *monomial* or *binomial* radical expression of the second order.

The number by which numerator and denominator are multiplied is called the *rationalizing factor.*

For a monomial divisor, $\sqrt{x}$, it is $\sqrt{x}$ itself. For a binomial divisor, $\sqrt{x} \pm \sqrt{y}$, it is the same binomial with the opposite sign, $\sqrt{x} \mp \sqrt{y}$.

EXERCISES

Reduce each of the following to equivalent fractions having a rational denominator.

1. $\dfrac{3}{2-\sqrt{5}}.$

2. $\dfrac{7}{\sqrt{5}+\sqrt{3}}.$

3. $\dfrac{\sqrt{27}}{\sqrt{3}+\sqrt{11}}.$

4. $\dfrac{2-\sqrt{7}}{2+\sqrt{7}}.$

5. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}.$

6. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}.$

7. $\dfrac{3\sqrt{3}-2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}.$

8. $\dfrac{\sqrt{a^2+1}-\sqrt{a^2-1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}.$

9. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}.$

10. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}.$

193. In finding the value of such an expression as $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, the approximation of *two* square roots and division by a decimal fraction would be required. But $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$ equals $\frac{11+2\sqrt{21}}{4}$ which requires only *one* root and division by the integer 4.

EXERCISES

Find the approximate values of the following expressions to three places of decimals.

1. $\dfrac{3\sqrt{5}+4\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

2. $\dfrac{\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.

3. $\dfrac{4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

4. $\dfrac{11\sqrt{5}-3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.

5. $\dfrac{7\sqrt{5}+3\sqrt{8}}{2\sqrt{5}-3\sqrt{2}}$.

6. $\dfrac{5\sqrt{19}-3\sqrt{7}}{3\sqrt{7}-\sqrt{19}}$.

7. $\dfrac{3\sqrt{2}-\sqrt{5}}{\sqrt{5}-6\sqrt{2}}$.

8. $\dfrac{5\sqrt{6}-7\sqrt{13}}{3\sqrt{13}-7\sqrt{6}}$.

194. Square root of a radical expression. A radical expression of the second order is sometimes a perfect square, and its **square root** may be written by inspection.

E.g. The square of $\sqrt{a} \pm \sqrt{b}$ is $a + b \pm 2\sqrt{ab}$. Hence if a radical expression can be put into the form $x \pm 2\sqrt{y}$, where x is the *sum* of two numbers a and b whose *product* is y, then $\sqrt{a} \pm \sqrt{b}$ is the *square root* of $x + 2\sqrt{y}$.

Example. Find the square root of $5 + \sqrt{24}$.

Since $5 + \sqrt{24} = 5 + 2\sqrt{6}$, in which 5 is the *sum* of 2 and 3, and 6 is their *product*, we have $\sqrt{5 + \sqrt{24}} = \sqrt{2} + \sqrt{3}$.

EXERCISES

Find the square root of each of the following:

1. $3 - 2\sqrt{2}$.
2. $7 + \sqrt{40}$.
3. $8 - \sqrt{60}$.
4. $7 + 4\sqrt{3}$.
5. $24 - 6\sqrt{7}$.
6. $28 + 3\sqrt{12}$.

195. Radical expressions involving imaginaries. According to the definition, § 112, $(\sqrt{-1})^2 = -1$. Hence, $(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -\sqrt{-1}$ and $(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1$.

The following examples illustrate operations with radical expressions containing imaginaries.

Ex. 1. $\sqrt{-4} + \sqrt{-9} = \sqrt{4}\sqrt{-1} + \sqrt{9}\sqrt{-1}$
$= (2+3)\sqrt{-1} = 5\sqrt{-1}$.

Ex. 2. $\sqrt{-4}\cdot\sqrt{-9} = \sqrt{4}\cdot\sqrt{9}\cdot(\sqrt{-1})^2 = -2\cdot 3 = -6$.

Ex. 3. $\dfrac{\sqrt{-4}}{\sqrt{-9}} = \dfrac{\sqrt{4}\sqrt{-1}}{\sqrt{9}\sqrt{-1}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$.

Ex. 4. $\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-6} = \sqrt{2}\cdot\sqrt{3}\cdot\sqrt{6}\cdot(\sqrt{-1})^3$
$= -\sqrt{36}\sqrt{-1} = -6\sqrt{-1}$.

Ex. 5. Simplify $(\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3$.

We are to use $\frac{1}{2}(1 + \sqrt{3}\sqrt{-1})$ three times as a factor. Reserving $(\frac{1}{2})^3 = \frac{1}{8}$ as the final coefficient, we have,

$$\begin{array}{l}
1 + \sqrt{3}\sqrt{-1} \\
\underline{1 + \sqrt{3}\sqrt{-1}} \\
1 + \sqrt{3}\sqrt{-1} \\
\underline{\qquad \sqrt{3}\sqrt{-1} - 3} \\
1 + 2\sqrt{3}\sqrt{-1} - 3
\end{array}
\qquad\qquad
\begin{array}{l}
-2 + 2\sqrt{3}\sqrt{-1} \\
\underline{\quad 1 + \sqrt{3}\sqrt{-1}} \\
-2 + 2\sqrt{3}\sqrt{-1} \\
\underline{\qquad -2\sqrt{3}\sqrt{-1} - 6} \\
-2 \qquad\qquad\qquad -6 = -8.
\end{array}$$

Hence $(\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3 = \frac{1}{8}(-8) = -1$.

EXERCISES

Perform the following indicated operations.

1. $\sqrt{-16}+\sqrt{-9}+\sqrt{-25}$.

2. $\sqrt{-x^4}-\sqrt{-x^2}$.

3. $3+5\sqrt{-1}-2\sqrt{-1}$.

4. $(2+3\sqrt{-1})(3+2\sqrt{-1})$.

5. $(2+3\sqrt{-1})(2-3\sqrt{-1})$.

6. $(4+5\sqrt{-3})(4-5\sqrt{-3})$.

7. $(2\sqrt{2}-3\sqrt{-3})(3\sqrt{3}+2\sqrt{-2})$.

8. $(\sqrt{-3}+\sqrt{-2})(\sqrt{-3}-\sqrt{-2})$.

9. $(3\sqrt{5}+2\sqrt{-7})(2\sqrt{5}-3\sqrt{-7})$.

10. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.

Rationalize the denominators of

11. $\dfrac{2}{1-\sqrt{-1}}$.

12. $\dfrac{3}{\sqrt{3}+\sqrt{-1}}$.

13. $\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}$.

14. $\dfrac{\sqrt{2}+\sqrt{-3}}{\sqrt{2}-\sqrt{-3}}$.

15. $\dfrac{5}{2-3\sqrt{-5}}$.

16. $\dfrac{x+y\sqrt{-1}}{x\sqrt{-1}-y}$.

17. Solve $x^4-1=0$ by factoring. Find four roots and verify each.

18. Solve $x^3+1=0$ by factoring and the quadratic formula. Find three roots and verify each.

19. Solve $x^3-1=0$ as in the preceding and verify each root.

20. Solve $x^6-1=0$ by factoring and the quadratic formula.

SOLUTION OF EQUATIONS CONTAINING RADICALS

196. Many equations containing radicals are reducible to equivalent rational equations of the first or second degree.

The method of solving such equations is shown in the following examples.

Ex. 1. Solve $$1+\sqrt{x}=\sqrt{3+x}. \quad (1)$$

Squaring and transposing, $2\sqrt{x}=2. \quad (2)$

Dividing by 2 and squaring, $x=1. \quad (3)$

Substituting in (1), $1+1=\sqrt{3+1}=2.$

Observe that only principal roots are used in this example.

If (1) is written $$1+\sqrt{x}=-\sqrt{3+x}, \quad (4)$$

then (2) and (3) follow as before, but $x=1$ does *not satisfy* (4). Indeed equation (4) has no root. This should not be confused with the fact that every integral, rational equation has at least one root.

Ex. 2. Solve $$\sqrt{x+5}=x-1. \quad (1)$$

Squaring and transposing, $x^2-3x-4=0. \quad (2)$

Solving, $x=4$ and $x=-1$.

$x=4$ satisfies (1) if the *principal* root in $\sqrt{x+5}$ is taken. $x=-1$ does not satisfy (1) as it stands but would if the *negative* root were taken.

Ex. 3. Solve $$\frac{\sqrt{4x+1}-\sqrt{3x-2}}{\sqrt{4x+1}+\sqrt{3x-2}}=\frac{1}{5}. \quad (1)$$

Clearing of fractions and combining similar radicals.

$$2\sqrt{4x+1}=3\sqrt{3x-2}. \quad (2)$$

Squaring and solving, we find $x=2$.

This value of x satisfies (1) when *all* the roots are taken *positive* and also when all are taken *negative*, but otherwise *not*.

Ex. 4. Solve $\sqrt{2x+3}=\dfrac{3x-1}{\sqrt{3x}-1}-1.$ (1)

The fraction in the second member should be reduced as follows:

$$\frac{3x-1}{\sqrt{3x}-1}=\frac{(\sqrt{3x}-1)(\sqrt{3x}+1)}{\sqrt{3x}-1}=\sqrt{3x}+1.$$

Hence, (1) reduces to $\sqrt{2x+3}=\sqrt{3x}+1-1=\sqrt{3x}.$ (2)

Solving, $x=3$, which satisfies (1).

If we clear (1) of fractions in the ordinary manner, we have

$$(\sqrt{3x}-1)\sqrt{2x+3}=-\sqrt{3x}+3x. \tag{2'}$$

Squaring both sides and transposing all rational terms to the second member,

$$2x\sqrt{3x}-6\sqrt{3x}=3x^2-8x-3. \tag{3}$$

Factoring each member,

$$2(x-3)\sqrt{3x}=(x-3)(3x+1), \tag{4}$$

which is satisfied by $x=3.$

Dividing each member by $x-3$, squaring and transposing, we have

$$9x^2-6x+1=(3x-1)^2=0, \tag{5}$$

which is satisfied by $x=\frac{1}{3}.$

Equation (1) is *not* satisfied by $x=\frac{1}{3}$, since the fraction in the second member is reduced to $\frac{0}{0}$ by this substitution. See § 50. The root $x=\frac{1}{3}$ is *introduced by clearing of fractions without first reducing the fraction to its lowest terms*, $\sqrt{3x}-1$ being a factor of both the numerator and the denominator. See § 165.

Ex. 5. Solve $\dfrac{6-x}{\sqrt{6-x}}-\sqrt{3}=\dfrac{x-3}{\sqrt{x-3}}.$ (1)

Reducing the fractions by removing common factors, we have

$$\sqrt{6-x}-\sqrt{3}=\sqrt{x-3}. \tag{2}$$

Squaring, transposing, and squaring again,

$$x^2-9x+18=0, \tag{3}$$

whence $x=3,\ x=6.$

But neither of these is a root of (1). In this case (1) has *no* root.

197. In solving an equation containing radicals, we note the following:

(1) If a fraction of the form $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$ is involved, this should be reduced by dividing numerator and denominator by $\sqrt{a}-\sqrt{b}$ *before* clearing of fractions.

(2) After clearing of fractions, transpose terms so as to leave one radical alone in one member.

(3) Square both members, and if the resulting equation still contains radicals, transpose and square as before.

(4) In every case verify all results by substituting in the given equation. In case any value does not satisfy the given equation, determine whether the roots could be so taken that it would. See Ex. 3.

EXERCISES

Solve the following equations:

1. $\sqrt{x^2+7x-2}-\sqrt{x^2-3x+6}=2.$

2. $\sqrt{3y}-\sqrt{3y-7}=\dfrac{5}{\sqrt{3y-7}}.$

3. $\dfrac{by-1}{\sqrt{by}+1}=\dfrac{\sqrt{by}-1}{2}+4.$

4. $\sqrt{5x-19}+\sqrt{3x+4}=9.$

5. $\dfrac{\sqrt{x^2+a^2}-x}{\sqrt{x^2+a^2}+x}=2.$

6. $\sqrt{a+\sqrt{ax+x^2}}=\sqrt{a}-\sqrt{x}.$

7. $\dfrac{y-l}{\sqrt{y}+\sqrt{l}}=\dfrac{\sqrt{y}-\sqrt{l}}{3}+2\sqrt{l}.$

8. $\sqrt{5x}+1=1-\dfrac{5x-1}{\sqrt{5x}+1}.$

9. $\dfrac{4}{x}-\dfrac{\sqrt{4-x^2}}{x}=\sqrt{3}.$

10. $\dfrac{4+x+\sqrt{8x+x^2}}{4+x-\sqrt{8x+x^2}}=4.$

11. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x+a}{\sqrt{x+a}}=\sqrt{a-b}.$

12. $\dfrac{x-a}{\sqrt{x}-\sqrt{a}}=\dfrac{\sqrt{x}+\sqrt{a}}{2}+2\sqrt{a}.$

13. $\dfrac{m-y}{\sqrt{m-y}}-\sqrt{m-n}=-\dfrac{y-n}{\sqrt{y-n}}.$

14. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}.$

15. $\sqrt{2x+7}+\sqrt{2x+14}=\sqrt{4x+35+2\sqrt{4x^2+42x-21}}.$

16. $\sqrt{x-3}+\sqrt{x+9}=\sqrt{x+18}+\sqrt{x-6}.$

17. $\sqrt{x+7}-\sqrt{x-1}=\sqrt{x+2}+\sqrt{x-2}.$

18. $a\sqrt{y+b}-c\sqrt{b-y}=\sqrt{b(a^2+c^2)}.$

19. $y\sqrt{y-c}-\sqrt{y^3+c^3}+c\sqrt{y+c}=0.$

20. $\dfrac{\sqrt{x}}{\sqrt{m}}+\dfrac{\sqrt{m}}{\sqrt{x}}=\dfrac{\sqrt{m}}{\sqrt{n}}+\dfrac{\sqrt{n}}{\sqrt{m}}.$

21. $\sqrt{14+\sqrt{x}}+\sqrt{6-\sqrt{x}}=\dfrac{12}{\sqrt{6-\sqrt{x}}}.$

22. $\sqrt{3x}+\sqrt{3x+13}=\dfrac{91}{\sqrt{3x+13}}.$

23. $\sqrt{6x+3}+\sqrt{x+3}=2x+3.$

24. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}.$

25. $\dfrac{\sqrt{x-a}+\sqrt{x-b}}{\sqrt{x-a}-\sqrt{x-b}}=\sqrt{\dfrac{x-a}{x-b}}.$

26. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$

27. $\sqrt{2x+2}+\sqrt{7+6x}=\sqrt{7x+72}.$

28. $\sqrt{2abx}+\sqrt{a^2-bx}=\sqrt{a^2+bx}.$

29. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{c}{x}.$

30. $\sqrt{x^2-2x+4}+\sqrt{3x^2+6x+12}=2\sqrt{x^2+x+10}.$

PROBLEMS

1. Find the altitude drawn to the longest side of the triangle whose sides are 6, 7, 8.

HINT. See figure, p. 235, E. C. Calling x and $8 - x$ the segments of the base and h the altitude, set up and solve two equations involving x and h.

2. Find the area of a triangle whose sides are 15, 17, 20.

First find one altitude as in problem 1.

3. Find the area of a triangle whose base is 16 and whose sides are 10 and 14.

4. Find the altitude on a side a of a triangle two of whose sides are a and a third b.

A three-sided pyramid all of whose edges are equal is called a regular tetrahedron. In Figure 10 AB, AC, AD, BC, BD, CD are all equal.

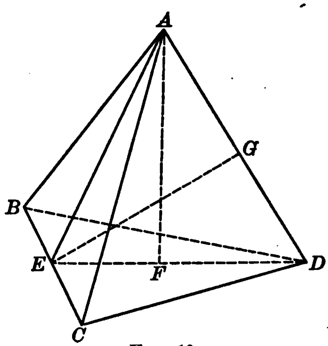


FIG. 10.

5. Find the altitude of a regular tetrahedron whose edges are each 6. Also the area of the base.

HINT. First find the altitudes AE and DE and then find the altitude of the triangle AED on the side DE, *i.e.* find AF.

6. Find the volume of a regular tetrahedron whose edges are each 10.

The volume of a tetrahedron is $\frac{1}{3}$ the product of the base and the altitude.

7. Find the volume of a regular tetrahedron whose edges are a.

8. In Figure 10 find EG if the edges are a.

9. If in Figure 10 EG is 12, compute the volume.

Use problem 8 to find the edge, then use problem 7 to find the volume.

10. Express the volume of the tetrahedron in terms of EG. That is if $EG=b$, find a general expression for the volume in terms of b.

11. If the altitude of a regular tetrahedron is 10, compute the edge accurately to two places of decimals.

12. Express the edge of a regular tetrahedron in terms of its altitude.

13. Express the volume of a regular tetrahedron in terms of its altitude.

14. Express the edge of a regular tetrahedron in terms of its volume.

15. Express the altitude of a regular tetrahedron in terms of its volume.

16. Express EG of Figure 10 in terms of the volume of the tetrahedron.

17. Find the edge of a regular tetrahedron such that its volume multiplied by $\sqrt{2}$, plus its entire surface multiplied by $\sqrt{3}$, is 144.

The resulting equation is of the third degree. Solve by factoring.

18. An electric light of 32 candle power is 25 feet from a lamp of 6 candle power. Where should a card be placed between them so as to receive the same amount of light from each?

Compare problem 13, p. 441. Compute result accurately to two places of decimals.

19. Where must the card be placed in problem 18 if the lamp is between the card and the electric light?

Notice that the roots of the equations in 18 and 19 are the same. Explain what this means.

20. State and solve a general problem of which 18 and 19 are special cases.

21. If the distance between the earth and the sun is 93 million miles, and if the mass of the sun is 300,000 times that of the earth, find two positions in which a particle would be equally attracted by the earth and the sun.

The gravitational attraction of one body upon another varies *inversely* as the square of the distance and directly as the product of the masses. Represent the mass of the earth by unity.

22. Find the volume of a pyramid whose altitude is 7 and whose base is a regular hexagon whose sides are 7.

The volume of a pyramid or a cone is $\frac{1}{3}$ the product of its base and its altitude.

23. If the volume of the pyramid in problem 22 were 100 cubic inches, what would be its altitude, a side of the base and the altitude being equal? Approximate the result to two places of decimals.

24. Express the altitude of the pyramid in problem 22 in terms of its volume, the altitude and the sides of the base being equal.

25. If in a right prism the altitude is equal to a side of the base, find the volume, the base being an equilateral triangle whose sides are a.

The volume of a right prism or cylinder equals the product of its base and its altitude.

26. Find the volume of the prism in problem 25 if its base is a regular hexagon whose side is a.

27. Express the side of the base of the prism in problem 25 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

28. Express the side of the base of the prism in problem 26 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

In Figures 11 and 12 the altitudes are each supposed to be three times the side a of the regular hexagonal bases.

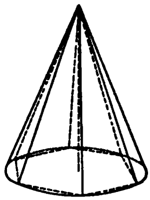


FIG. 11.

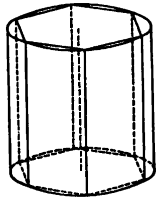


FIG. 12.

29. Express the difference between the volumes of the pyramid and the circumscribed cone in terms of a.

The volume of a cone equals $\frac{1}{3}$ the product of its base and altitude.

30. Express a in terms of the difference between the volumes of the cone and pyramid. State and solve a particular problem by means of the formula thus obtained.

31. Express the volume of the pyramid in terms of the difference between the areas of the bases of the cone and the pyramid. State a particular case and solve by means of the formula first obtained.

The lateral area of a right cylinder or prism equals the perimeter of the base multiplied by the altitude.

32. Express the difference of the lateral areas of the cylinder and the prism in terms of a.

The following four problems refer to Figure 12. In each case state a particular problem and solve by means of the formula obtained.

33. Express a in terms of the difference of the lateral areas.

34. Express the volume of the prism in terms of the difference of the perimeters of the bases.

35. Express the volume of the cylinder in terms of the difference of the lateral areas.

36. Express the sum of the volumes of the prism and cylinder in terms of the difference of the areas of the bases.

CHAPTER XI

LOGARITHMS

198. The operations of multiplication, division, and finding powers and roots are greatly shortened by the use of **logarithms.**

The logarithm of a number, in the system commonly used, is the *index of that power of* 10 *which equals the given number.*

Thus, 2 is the logarithm of 100 since $10^2 = 100$.

This is written $\log 100 = 2.$

Similarly $\log 1000 = 3$, since $10^3 = 1000$,

and $\log 10000 = 4$, since $10^4 = 10000$.

The logarithm of a number which is *not an exact* rational *power* of 10 is an irrational number and is written approximately as a decimal fraction.

Thus, $\log 74 = 1.8692$ since $10^{1.8692} = 74$ approximately.

In higher algebra it is shown that the laws for rational exponents (§ 179) hold also for irrational exponents.

199. The decimal part of a logarithm is called the **mantissa,** and the integral part the **characteristic.**

Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., it follows that for all numbers between 1 and 10 the logarithm lies between 0 and 1, that is, the characteristic is 0. Likewise for numbers between 10 and 100 the characteristic is 1, for numbers between 100 and 1000 it is 2, etc.

200. Tables of logarithms (see p. 470) usually give the mantissas only, the characteristics being supplied, in the case of *whole* numbers, according to § 199, and in the case of decimal numbers, as shown in the examples given under § 201.

201. An important property of logarithms is illustrated by the following:

From the table of logarithms, p. 470, we have:

$$\log 376 = 2.5752, \text{ or } 376 = 10^{2.5752}. \qquad (1)$$

Dividing both members of (1) by 10 we have

$$37.6 = 10^{2.5752} \div 10^1 = 10^{2.5752-1} = 10^{1.5752}.$$

Hence, $\log 37.6 = 1.5752,$

Similarly, $\log 3.76 = 1.5752 - 1 = 0.5752,$

$$\log .376 = 0.5752 - 1, \text{ or } \bar{1}.5752,$$

$$\log .0376 = 0.5752 - 2, \text{ or } \bar{2}.5752,$$

where $\bar{1}$ and $\bar{2}$ are written for -1 and -2 to indicate that the characteristics are negative while the mantissas are positive.

Multiplying (1) by 10 gives

$$\log 3760 \;= 2.5752 + 1 = 3.5752,$$

and $\log 37600 = 2.5752 + 2 = 4.5752.$

Hence, if the decimal point of a number is moved a certain number of places to the *right* or to the *left*, the characteristic of the logarithm is *increased* or *decreased* by a corresponding number of units, the mantissa *remaining the same.*

From the table on pp. 470, 471, we may find the mantissas of logarithms for all integral numbers from 1 to 1000. In this table the logarithms are given to four places of decimals, which is sufficiently accurate for most practical purposes.

E.g. for log 4 the mantissa is the same as that for log 40 or for log 400.

To find log .0376 we find the mantissa corresponding to 376, and prefix the characteristic $\bar{2}$. See above.

Ex. 1. Find log 876.

Solution. Look down the column headed *N* to 87, then along this line to the column headed 6, where we find the number 9425, which is the mantissa. Hence $\log 876 = 2.9425$.

Ex. 2. Find $\log 3747$.

Solution. As above we find $\log 3740 = 3.5729$,

and $\log 3750 = 3.5740$.

The difference between these logarithms is 11, which corresponds to a difference of 10 between the numbers. But 3740 and 3747 differ by 7. Hence, their logarithms should differ by $\frac{7}{10}$ of 11, *i.e.* by 8.1. Adding this to the logarithm of 3740, we have 3.5737, which is the required logarithm.

The assumption here made, that the logarithm varies directly as the number, is not quite, but very nearly, accurate, when the variation of the number is confined to a narrow range, as is here the case.

Ex. 3. Find the number whose logarithm is 2.3948.

Solution. Looking in the table, we find that the nearest *lower* logarithm is 2.3945 which corresponds to the number 248. See § 199.

The given mantissa is 3 greater than that of 248, while the mantissa of 249 is 17 greater. Hence the number corresponding to 2.3948 must be 248 plus $\frac{3}{17}$ or .176. Hence, 248.18 is the required number, correct to 2 places of decimals.

Ex. 4. Find $\log .043$.

Solution. Find $\log 43$ and subtract 3 from the characteristic.

Ex. 5. Find the number whose logarithm is $\bar{4}.3949$.

Solution. Find the number whose logarithm is 0.3949, and move the decimal point 4 places to the left.

EXERCISES

Find the logarithms of the following numbers:

1. 491.	6. .541.	11. .006.	16. 79.31.
2. 73.5.	7. .051.	12. .1902.	17. 4.245.
3. 2485.	8. 8104.	13. .0104.	18. .0006.
4. 539.7.	9. 70349.	14. 2.176.	19. 3.817.
5. 53.27.	10. 439.26.	15. 8.094.	20. .1341.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	7857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Find the numbers corresponding to the following logarithms:

21. 1.3179.	26. 2.9900.	31. $\bar{1}.5972$.	36. 0.2468.
22. 3.0146.	27. 0.1731.	32. $\bar{1}.0011$.	37. 0.1357.
23. 0.7145.	28. 0.8974.	33. $\bar{2}.7947$.	38. $\bar{2}.0246$.
24. $\bar{1}.5983$.	29. 0.9171.	34. $\bar{2}.5432$.	39. $\bar{1}.1358$.
25. 2.0013.	30. 3.4015.	35. 0.5987.	40. $\bar{4}.0478$.

202. Products and powers may be found by means of logarithms, as shown by the following examples.

Ex. 1. Find the product $49 \times 134 \times .071 \times 349$.

Solution. From the table,

$$\log 49 = 1.6902 \text{ or } 49 = 10^{1.6902}.$$

$$\log 134 = 2.1271 \text{ or } 134 = 10^{2.1271}.$$

$$\log .071 = \bar{2}.8513 \text{ or } .071 = 10^{\bar{2}.8513}.$$

$$\log 349 = 2.5428 \text{ or } 349 = 10^{2.5428}.$$

Since powers of the same base are multiplied by *adding* exponents, § 176, we have $49 \times 134 \times .071 \times 349 = 10^{5.2114}$.

Hence $\log (49 \times 134 \times .071 \times 349) = 5.2114$.

The number corresponding to this logarithm, as found by the method used in Ex. 3 above, is 162704. By actual multiplication the product is found to be 162698.914 or 162699 which is the nearest approximation without decimals. Hence the product obtained by means of logarithms is 5 too large. This is an error of about $\frac{1}{32000}$ of the actual result and is therefore so small as to be negligible.

Ex. 2. Find $(1.05)^{20}$.

Solution. $\text{Log } 1.05 = 0.0212$ or $10^{0.0212} = 1.05$.

Hence $(1.05)^{20} = (10^{0.0212})^{20} = 10^{(0.0212)\cdot 20} = 10^{0.424}$,

or $\log (1.05)^{20} = 0.4240$.

Hence $(1.05)^{20}$ is the number corresponding to the logarithm 0.4240.

Since logarithms are **exponents** of the base 10, it follows from the laws of exponents (see § 198) that

(*a*) *The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers.*

(*b*) *The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

That is,

$$\log(a \cdot b \cdot c) = \log a + \log b + \log c, \text{ and } \log a^n = n \log a.$$

EXERCISES

By means of the logarithms obtain the following products and powers:

1. $243 \times 76 \times .34$.
2. 823.68×370.
3. 216.83×2.03.
4. $57^2 \times (.71)^2$.
5. $510 \times (9.1)^3$.
6. $43.71 \times (21)^2$.
7. 5.93×10.02.
8. 486×3.45.
9. $(.02)^2 \times (0.8)$.
10. $(65)^2 \times (91)^3$.
11. $(84)^2 \times (75)^3$.
12. $(.960)^2(49)^2$.
13. $(49)^3 \times .19 \times 21^2$.
14. $.21084 \times (.53)^2$.
15. $7.865 \times (.013)^2$.
16. $(6.75)^3 \times (723)^2$.
17. $(1.46)^2 \times (61.2)^2$.
18. $(3.54)^3 \times (29.3)^2$.
19. $(4.132)^2 \times (5.184)^2$.
20. $1946 \times 398 \times .08$.

203. Quotients and roots may be found by means of logarithms, as shown by the following examples.

Ex. 1. Divide 379 by 793.

Solution. From the table,

$$\log 379 = 2.5786 \text{ or } 10^{2.5786} = 379.$$

$$\log 793 = 2.8993 \text{ or } 10^{2.8993} = 793.$$

Hence by the law of exponents for division, § 175,

$$379 \div 793 = 10^{2.5786 - 2.8993}.$$

Since in all operations with logarithms the mantissa is positive, write the first exponent 3.5786 − 1 and then subtract 2.8993.

Hence $\log(379 \div 793) = .6793 - 1 = \bar{1}.6793$.

Hence the quotient is the number corresponding to this logarithm.

Ex. 2. By means of logarithms approximate $\sqrt[3]{42^2 \times 37^5}$.

By the methods used above we find

$$\log (42^2 \times 37^5) = 11.0874 \text{ or } 10^{11.0874} = 42^2 \times 37^5.$$

Hence $$\sqrt[3]{42^2 \times 37^5} = (10^{11.0874})^{\frac{1}{3}} = 10^{\frac{11.0874}{3}} = 10^{3.6958}.$$

That is, $$\log \sqrt[3]{42^2 \times 37^5} = 3.6958.$$

Hence the result sought is the number corresponding to this logarithm.

It follows from the laws of exponents (see § 198) that

(*a*) *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

(*b*) *The logarithm of a root of a number is the logarithm of the number divided by the index of the root.*

That is

$$\log \frac{a}{b} = \log a - \log b \text{ and } \log \sqrt[n]{a} = \frac{\log a}{n}.$$

EXERCISES

By means of logarithms approximate the following quotients and roots:

1. $45.2 \div 8.9$.
2. $231.18 \div 4.2$.
3. $.04905 \div .327$.
4. $\sqrt{196 \times 256}$.
5. $\dfrac{5334 \times .02374}{27.43 \times 3.246}$.
6. $\sqrt[5]{69} \div \sqrt[3]{21}$.
7. $\sqrt[7]{15} \times \sqrt[8]{67}$.
8. $\sqrt[10]{211} \times \sqrt[11]{34.7}$.
9. $(5184)^{\frac{1}{2}} \div (38124)^{\frac{1}{3}}$.
10. $(6.75)^3 \div (2.132)^2$.
11. $\sqrt[9]{105} \div \sqrt[13]{76}$.
12. $(91125)^{\frac{1}{3}} \div (576)^{\frac{1}{3}}$.
13. $(3.040)^3 \div (.0065)^3$.
14. $(29.3)^{\frac{1}{3}} \div \sqrt{(3.47)^3}$.
15. $\sqrt[3]{39} \times \sqrt[3]{56} \times \sqrt[4]{87}$.
16. $\sqrt[3]{\dfrac{13^4 \times .31^2 \times 4.31^3}{\sqrt{71} \times \sqrt[3]{41} \times \sqrt{51}}}$.
17. $\sqrt[5]{\dfrac{4^9 \times .57^3 \times 42^3}{\sqrt[3]{5.2} \times \sqrt[5]{.83} \times \sqrt{23}}}$.
18. $\left(\dfrac{\sqrt[3]{54} \times \sqrt[4]{28} \times \sqrt[5]{7}}{\sqrt[2]{4^7} \times \sqrt[3]{7^4} \times (.003)^{\frac{3}{4}}}\right)^{\frac{4}{5}}$.

CHAPTER XII

PROGRESSIONS

ARITHMETIC PROGRESSIONS

204. An arithmetic progression is a series of numbers, such that any one after the first is obtained by adding a fixed number to the preceding. The fixed number is called the **common difference.**

The general form of an arithmetic progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,$$

where a is the first term and d the common difference.

E.g. 2, 5, 8, 11, 14, ⋯ is an arithmetic progression in which 2 is the first term and 3 the common difference. Written in the general form, it would be

$$2,\ 2+3,\ 2+2\cdot 3,\ 2+3\cdot 3,\ 2+4\cdot 3,\ \cdots.$$

205. If there are n terms in the progression, then the last term is $a+(n-1)d$. Indicating the last term by l, we have

$$l = a+(n-1)d. \qquad \text{I}$$

An arithmetic progression of n terms would then be written in general form, thus,

$$a,\ a+d,\ a+2d,\ \cdots,\ a+(n-2)d,\ a+(n-1)d.$$

EXERCISES

1. Solve I for each letter in terms of all the others.

In each of the following find the value of the letter not given, and write out the progression in each case.

2. $\begin{cases} a=2, \\ d=2, \\ n=7. \end{cases}$ **3.** $\begin{cases} a=3, \\ d=5, \\ l=43. \end{cases}$ **4.** $\begin{cases} a=1, \\ n=15, \\ l=15. \end{cases}$ **5.** $\begin{cases} a=7, \\ n=31, \\ l=91. \end{cases}$

6. $\begin{cases} a=4, \\ d=-3, \\ n=18. \end{cases}$ 8. $\begin{cases} a=3, \\ d=-5, \\ l=-32. \end{cases}$ 10. $\begin{cases} d=-5, \\ n=13, \\ l=-63. \end{cases}$ 12. $\begin{cases} a=11, \\ l=-39, \\ d=-5. \end{cases}$

7. $\begin{cases} a=-5, \\ d=4, \\ n=7. \end{cases}$ 9. $\begin{cases} d=7, \\ n=8, \\ l=24. \end{cases}$ 11. $\begin{cases} a=-3, \\ n=9, \\ l=-27. \end{cases}$ 13. $\begin{cases} a=x, \\ l=y, \\ n=z. \end{cases}$

206. The sum of an arithmetic progression of n terms may be obtained as follows:

Let s_n denote the sum of n terms of the progression. Then,

$$s_n = a + [a+d] + [a+2d] + \cdots + [a+(n-2)d] + [a+(n-1)d]. \quad (1)$$

This may also be written, reversing the order of the terms, thus,

$$s_n = [a+(n-1)d] + [a+(n-2)d] + \cdots + [a+2d] + [a+d] + a. \quad (2)$$

Adding (1) and (2), we have

$$2s_n = [2a+(n-1)d] + [2a+(n-2)d+d] + \cdots + [2a+(n-2)d+d] + [2a+(n-1)d].$$

The expression in each bracket is reducible to $2a+(n-1)d$, which may also be written $a+[a+(n-1)d] = a+l$, by § 205.

Since there are n of these expressions, each $a+l$, we have

$$2s_n = n(a+l).$$

Hence
$$s_n = \frac{n}{2}(a+l). \qquad \text{II}$$

This formula for the sum of n terms involves a, l, and n, that is, the first term, the last term, and the number of terms.

207. In the two equations,

$$l = a + (n-1)d, \qquad \text{I}$$

$$s = \frac{n}{2}(a+l), \qquad \text{II}$$

there are five letters, namely, a, d, l, n, s. If any three of these are given, the equations I and II may be solved simultaneously to find the other two, considered as the *unknowns*.

The solution of problems in arithmetic progression by means of equations I and II is illustrated in the following examples:

Ex. 1. Given $n = 11$, $l = 23$, $s = 143$. Find a and d.

Substituting the given values in I and II,

$$23 = a + (11 - 1)d. \qquad (1)$$

$$143 = \tfrac{11}{2}(a + 23). \qquad (2)$$

From (2), $a = 3$, which in (1) gives $d = 2$.

Ex. 2. Given $d = 4$, $n = 5$, $s = 75$. Find a and l.

From I and II, $$l = a + (5 - 1)4, \qquad (1)$$

$$75 = \frac{5}{2}(a + l). \qquad (2)$$

Solving (1) and (2) simultaneously, we have $a = 7$, $l = 23$.

Ex. 3. Given $d = 4$, $l = 35$, $s = 161$. Find a and n.

From I and II, $$35 = a + (n - 1)4, \qquad (1)$$

$$161 = \frac{n}{2}(a + 35). \qquad (2)$$

From (1) $$a = 39 - 4n,$$

which in (2) gives $$161 = \frac{n}{2}(74 - 4n) = 37n - 2n^2.$$

Hence $$n = \tfrac{23}{2}, \text{ or } 7.$$

Since an arithmetic progression must have an *integral* number of terms, only the second value is applicable to this problem.

Ex. 4. Given $d = 2$, $l = 11$, $s = 35$. Find a and n.

Substituting in I and II, and solving for a and n, we have

$$a = 3,\ n = 5, \text{ and } a = -1,\ n = 7.$$

Hence there are two progressions,

$$-1, 1, 3, 5, 7, 9, 11,$$

and $$3, 5, 7, 9, 11,$$

each of which satisfies the given conditions.

EXERCISES

In each of the following obtain the values of the two letters not given.

If fractional or negative values of n are obtained, such a result indicates that the problem is impossible. This is also the case if an *imaginary* value is obtained for *any* letter. In each exercise interpret all the values found.

1. $\begin{cases} s=96, \\ l=19, \\ d=2. \end{cases}$ 4. $\begin{cases} s=88, \\ l=-7, \\ d=-3. \end{cases}$ 7. $\begin{cases} d=-1, \\ n=41, \\ l=-35. \end{cases}$ 10. $\begin{cases} d=6, \\ l=49, \\ s=232. \end{cases}$

2. $\begin{cases} s=34, \\ l=14, \\ d=3. \end{cases}$ 5. $\begin{cases} n=18, \\ a=4, \\ l=13. \end{cases}$ 8. $\begin{cases} l=30, \\ s=162, \\ n=9. \end{cases}$ 11. $\begin{cases} s=7, \\ d=1\frac{1}{2}, \\ l=7. \end{cases}$

3. $\begin{cases} a=7, \\ l=27, \\ s=187. \end{cases}$ 6. $\begin{cases} n=14, \\ a=7, \\ s=14. \end{cases}$ 9. $\begin{cases} a=30, \\ n=10, \\ s=120. \end{cases}$ 12. $\begin{cases} s=14, \\ d=3, \\ l=4. \end{cases}$

In each of the following call the two letters specified the *unknowns* and solve for their values in terms of the remaining three letters supposed to be *known*.

13. a, d. **15.** a, n. **17.** d, l. **19.** d, s. **21.** l, s.

14. a, l. **16.** a, s. **18.** d, n. **20.** l, n. **22.** n, s.

208. **Arithmetic means.** The terms between the first and the last of an arithmetic progression are called **arithmetic means.**

Thus, in 2, 5, 8, 11, 14, 17, the four arithmetic means between 2 and 17 are 5, 8, 11, 14.

If the first and the last terms and the number of arithmetic means between them are given, then these means can be found.

For we have given a, l, and n. Hence d can be found and the whole series constructed.

Example. Insert 7 arithmetic means between 3 and 19.

In this progression $a = 3$, $l = 19$, and $n = 9$.

Hence from $l = a + (n-1)d$ we find $d = 2$ and the required means are 5, 7, 9, 11, 13, 15, 17.

209. The case of *one* arithmetic mean is important. Let A be the arithmetic mean between a and l. Since a, A, l are in arithmetic progression, we have $A = a + d$, and $l = A + d$. Hence

$$A - l = a - A$$

or

$$A = \frac{a+l}{2}. \qquad \text{III}$$

EXERCISES AND PROBLEMS

1. Insert 5 arithmetic means between 5 and -7.
2. Insert 3 arithmetic means between -2 and 12.
3. Insert 8 arithmetic means between -3 and -5.
4. Insert 5 arithmetic means between -11 and 40.
5. Insert 15 arithmetic means between 1 and 2.
6. Insert 9 arithmetic means between $2\frac{3}{4}$ and $-1\frac{1}{2}$.
7. Find the arithmetic mean between 3 and 17.
8. Find the arithmetic mean between -4 and 16.
9. Find the tenth and eighteenth terms of the series 4, 7, 10, ⋯.
10. Find the fifteenth and twentieth terms of the series $-8, -4, 0, \cdots$.
11. The fifth term of an arithmetic progression is 13 and the thirtieth term is 49. Find the common difference.
12. Find the sum of all the integers from 1 to 100.
13. Find the sum of all the odd integers between 0 and 200.
14. Find the sum of all integers divisible by 6 between 1 and 500.
15. Show that $1 + 3 + 5 + \cdots + n = n^2$ where n is any odd integer.

16. In a potato race 40 potatoes are placed in a straight line one yard apart, the first potato being two yards from the basket. How far must a contestant travel in bringing them to the basket one at a time?

17. There are three numbers in arithmetic progression whose sum is 15. The product of the first and last is $3\frac{1}{5}$ times the second. Find the numbers.

18. There are four numbers in arithmetic progression whose sum is 20 and the sum of whose squares is 120. Find the numbers.

19. If a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 the third, etc., how far will it fall in 10 seconds? 15 seconds? t seconds?

20. According to the law indicated in problem 19 in how many seconds will a body fall 1000 feet? s feet?

If a body is thrown downward with a velocity of v_0 feet per second, then the distance, s, it will fall in t seconds is v_0t feet plus the distance it would fall if starting from rest.

$$\text{That is, } s = v_0t + \tfrac{1}{2}gt^2, \text{ where } g = 32.16.$$

21. In what time will a body fall 1000 feet if thrown downward with a velocity of 20 feet per second?

22. With what velocity must a body be thrown downward in order that it shall fall 360 feet in 3 seconds?

23. A stone is dropped into a well, and the sound of its striking the bottom is heard in 3 seconds. How deep is the well if sound travels 1080 feet per second?

A body thrown upward with a certain velocity will rise as far as it would have to fall to acquire this velocity. The velocity (neglecting the resistance of the atmosphere) of a body starting from rest is gt where $g = 32.16$ and t is the number of seconds.

24. A rifle bullet is shot directly upward with a velocity of 2000 feet per second. How high will it rise, and how long before it will reach the ground?

25. From a balloon 5800 feet above the earth, a body is thrown downward with a velocity of 40 feet per second. In how many seconds will it reach the ground?

26. If in Problem 25 the body is thrown upward at the rate of 40 feet per second, how long before it will reach the ground?

GEOMETRIC PROGRESSIONS

210. A **geometric progression** is a series of numbers in which any term after the first is obtained by multiplying the preceding term by a fixed number, called the **common ratio.**

The general form of a geometric progression is

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},$$

in which a is the first term, r the constant multiplier, or common ratio, and n the number of terms.

E.g. 3, 6, 12, 24, 48, is a geometric progression in which 3 is the first term, 2 is the common ratio, and 5 is the number of terms.

Written in the general form it would be $3,\ 3\cdot 2,\ 3\cdot 2^2,\ 3\cdot 2^3,\ 3\cdot 2^4$.

211. If l is the last or nth term of the series, then

$$l = ar^{n-1}. \qquad \text{I}$$

If any three of the four letters in I are given, the remaining one may be found by solving this equation.

EXERCISES

In each of the following find the value of the letter not given:

1. $\begin{cases} l=162, \\ r=3, \\ n=5. \end{cases}$

2. $\begin{cases} a=1, \\ r=2, \\ n=8. \end{cases}$

3. $\begin{cases} a=-4, \\ r=-3, \\ n=6. \end{cases}$

4. $\begin{cases} a=-1, \\ r=-2, \\ n=9. \end{cases}$

5. $\begin{cases} l=1024, \\ r=-2, \\ n=11. \end{cases}$

6. $\begin{cases} l=1024, \\ r=2, \\ n=11. \end{cases}$

7. $\begin{cases} a=-\frac{1}{2}, \\ r=\frac{3}{2}, \\ n=6. \end{cases}$

8. $\begin{cases} l=18, \\ r=\frac{1}{3}, \\ n=6. \end{cases}$

9. $\begin{cases} l=-16, \\ r=-\frac{3}{4}, \\ n=5. \end{cases}$

10. $\begin{cases} l=32, \\ r=-2, \\ n=6. \end{cases}$

11. $\begin{cases} a=-2, \\ r=-\frac{3}{2}, \\ n=7. \end{cases}$

12. $\begin{cases} a=3, \\ r=2, \\ l=1536. \end{cases}$

212. The **sum of n terms** of a geometric expression may be found as follows:

If s_n denotes the sum of n terms, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying both members of (1) by r, we have

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Subtracting (1) from (2), and canceling terms, we have

$$rs_n - s_n = ar^n - a. \qquad (3)$$

Solving (3) for s_n we have

$$s_n = \frac{ar^n - a}{r-1} = \frac{a(r^n-1)}{r-1}. \qquad \text{II}$$

This formula for the sum of n terms of a geometric series involves only a, r, and n.

Since $ar^n = r \cdot ar^{n-1} = r \cdot l$, s^n may also be written:

$$s_n = \frac{rl - a}{r-1} = \frac{a - rl}{1-r}. \qquad \text{III}$$

This formula involves only r, l, and a.

213. From equations I and II or I and III any two of the numbers a, l, r, s, and n can be found when the other three are given, as in the following examples.

Ex. 1. Given $n = 7$, $r = 2$, $s = 381$. Find a and l.

From I and III, $\quad l = a \cdot 2^6 = 64\,a, \qquad (1)$

$$381 = \frac{2\,l - a}{2-1} = 2\,l - a. \qquad (2)$$

Substituting $l = 64\,a$ in (2), we obtain $a = 3$, and $l = 192$.

Ex. 2. Given $a = -3$, $l = -243$, $s = -183$. Find r and n.

From I and III, $\quad -243 = (-3)r^{n-1}, \qquad (1)$

$$-183 = \frac{-243\,r + 3}{r-1}. \qquad (2)$$

From (2) $\qquad r = -3. \qquad (3)$

From (1) $\qquad 81 = (-3)^{n-1}. \qquad (4)$

Since $(-3)^4 = 81$, we have $n - 1 = 4$ or $n = 5$.

EXERCISES

1. Solve II for a in terms of the remaining letters.

2. Solve III for each letter in terms of the remaining letters.

In each of the following find the terms represented by the interrogation points.

3. $\begin{cases} a = 1, \\ r = 3, \\ n = 5, \\ s = ? \end{cases}$ **4.** $\begin{cases} s = 635, \\ r = 2, \\ n = 7, \\ a = ? \end{cases}$ **5.** $\begin{cases} s = 13, \\ r = \frac{2}{3}, \\ n = 4, \\ a = ? \end{cases}$ **6.** $\begin{cases} l = -\frac{16}{81}, \\ s = ? \\ n = 5, \\ r = \frac{2}{3}. \end{cases}$

7. $\begin{cases} a = 1, \\ s = \frac{25}{64}, \\ l = -\frac{27}{64}, \\ r = ? \end{cases}$ **8.** $\begin{cases} r = \frac{1}{6}, \\ n = 5, \\ l = 1296, \\ a = ? \\ s = ? \end{cases}$ **9.** $\begin{cases} r = \frac{3}{2}, \\ n = 8, \\ s = 1050\frac{5}{6}, \\ l = ? \\ a = ? \end{cases}$ **10.** $\begin{cases} a = \frac{9}{2}, \\ n = 7, \\ l = \frac{32}{81}, \\ r = ? \\ s = ? \end{cases}$

214. Geometric means. The terms between the first and the last of a geometric progression are called **geometric means**.

Thus in 3, 6, 12, 24, 48, three geometric means between 3 and 48 are 6, 12 and 24.

If the first term, the last term, and the number of geometric means are given, the ratio may be found from I, and then the means may be inserted.

Example. Insert 4 geometric means between 2 and 64.

We have given $a = 2$, $l = 64$, $n = 4 + 2 = 6$, to find r.

From I, $\qquad 64 = 2 \cdot r^{6-1}$ or $r^5 = 32$ and $r = 2$.

Hence, the series is 2, 4, 8, 16, 32, 64.

215. The case of *one* geometric mean is important. If G is the geometric mean between a and b, we have $\dfrac{G}{a} = \dfrac{b}{G}$.

Hence, $\qquad \boldsymbol{G = \sqrt{ab}}.$

216. Problem. In attempting to reduce $\frac{2}{3}$ to a decimal, we find by division .666 ⋯, the dots indicating that the process goes on indefinitely.

Conversely, we see that $.666 \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$, that is, a geometric progression in which $a = \frac{6}{10}$, $r = \frac{1}{10}$, and n is not fixed but goes on *increasing indefinitely.*

As n grows large, l grows small, and by taking n sufficiently large, l can be made as small as we please. Hence formula III, § 212, is to be interpreted in this case as follows:

$$s_n = \frac{a - rl}{1 - r} = \frac{\frac{6}{10} - \frac{l}{10}}{1 - \frac{1}{10}} = \frac{6 - l}{9},$$

in which l grows small indefinitely as n increases indefinitely, so that by taking n large enough s_n can be made to differ as little as we please from $\frac{6-0}{9} = \frac{6}{9} = \frac{2}{3}$.

In this case we say s_n **approaches** $\frac{2}{3}$ **as a limit** as n increases indefinitely.

Observe that this interpretation can apply only when the constant multiplier r is a proper fraction.

EXERCISES AND PROBLEMS

1. Insert 5 geometric means between 2 and 128.
2. Insert 7 geometric means between 1 and $\frac{1}{256}$.
3. Find the geometric mean between 8 and 18.
4. Find the geometric mean between $\frac{1}{12}$ and $\frac{1}{4}$.
5. Find the fraction which is the limit of .333 ⋯.
6. Find the fraction which is the limit of .1666 ⋯.
7. Find the fraction which is the limit of .08333 ⋯.
8. Find the 13th term of $-\frac{16}{3}, 4, -3 \cdots$.
9. Find the sum of 15 terms of the series $-243, 81, -27 \cdots$.
10. Find the limit of the sum $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \cdots$, as the number of terms increases indefinitely.

Given	Find	Given	Find
11. a, r, n	l, s	15. a, n, l	s, r
12. a, r, s	l	16. r, n, l	s, a
13. r, n, s	l, a	17. r, l, s	a
14. a, r, l	s	18. a, l, s	r

19. The product of three terms of a geometric progression is 1000. Find the second term.

20. Four numbers are in geometric progression. The sum of the second and third is 18, and the sum of the first and fourth is 27. Find the numbers.

21. Find an arithmetic progression whose first term is 1 and whose first, second, fifth, and fourteenth terms are in geometric progression.

22. Three numbers whose sum is 27 are in arithmetic progression. If 1 is added to the first, 3 to the second, and 11 to the third the sums will be in geometric progression. Find the numbers.

23. To find the compound interest when the principal, the rate of interest, and the time are given.

Solution. Let p equal the number of dollars invested, r the rate of per cent of interest, t the number of years, and a the amount at the end of t years.

Then $a = p(1+r)$ at the end of one year.

$a = p(1+r)(1+r) = p(1+r)^2$ at the end of two years.

and $a = p(1+r)^t$ at the end of t years.

That is, the amount for t years is the last term of a geometric progression in which p is the first term, $1 + r$ is the ratio, and $t + 1$ is the number of terms.

24. Show how to modify the solution given under problem 23 when the interest is compounded semiannually; quarterly.

25. Solve the equation $a = p(1+r)^t$ for p and for r.

26. Solve $a = p(1+r)^t$ for t.

Solution. $\log a = \log p(1+r)^t = \log p + \log (1+r)^t$

$= \log p + t \log (1+r)$. (See § 202.) Hence $t = \dfrac{\log a - \log p}{\log (1+r)}$.

27. At what rate of interest compounded annually will \$1200 amount to \$1800 in 12 years?

28. At what rate of interest compounded semiannually will a sum double itself in 20 years? in 15 years? in 10 years?

29. In what time will \$8000 amount to \$13,500, the rate of interest being $3\frac{1}{2}\%$ compounded annually?

30. In what time will a sum double itself at 3%, 4%, 5%, compounded semiannually?

The present value of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

31. What is the present value of \$2500 due in 4 years, money being worth $3\frac{1}{2}\%$ interest compounded semiannually?

32. A man bequeathed \$50,000 to his daughter, payable on her twenty-fifth birthday, with the provision that the present worth of the bequest should be paid in case she married before that time. If she married at 21, how much would she receive, interest being 4% per annum and compounded quarterly?

33. What is the rate of interest if the present worth of \$24,000 due in 7 years is \$19,500?

34. In how many years is \$5000 due if its present worth is \$3500, the rate of interest being $3\frac{3}{4}\%$ compounded annually?

HARMONIC PROGRESSIONS

217. A **harmonic progression** is a series whose terms are the reciprocals of the corresponding terms of an arithmetic progression.

E.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} \cdots$ is a harmonic progression whose terms are the reciprocals of the terms of the arithmetic progression $1, 3, 5, 7, 9 \cdots$.

The name *harmonic* is given to such a series because musical strings of uniform size and tension, whose lengths are the reciprocals of the positive integers, *i.e.* $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ ..., vibrate in harmony.

The general form of the harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}. \qquad \text{I}$$

It follows that if $a, b, c, d, e, \ldots$ are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \cdots$ are in arithmetic progression. Hence, all questions pertaining to a harmonic progression are best answered by first converting it into an arithmetic progression.

218. Harmonic means. The terms between the first and the last of a harmonic progression are called **harmonic means** between them.

Example. Insert five harmonic means between 30 and 3.

This is done by inserting five arithmetic means between $\frac{1}{30}$ and $\frac{1}{3}$. By the method of § 208 the arithmetic series is found to be $\frac{1}{30}, \frac{1}{12}, \frac{2}{15}, \frac{11}{60}, \frac{7}{30}, \frac{17}{60}, \frac{1}{3}$. Hence, the harmonic series is $30, 12, \frac{15}{2}, \frac{60}{11}, \frac{30}{7}, \frac{60}{17}, 3$.

219. The case of a single harmonic mean is important. Let a, H, l be in harmonic progression. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{l}$ are in arithmetic progression.

Hence, by § 209, $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{l}}{2}$ or $H = \frac{2al}{a+l}$.

220. The arithmetic, geometric, and harmonic means between a and l are related as follows:

We have seen $\quad A = \frac{a+l}{2}, G = \sqrt{al}, H = \frac{2al}{a+l}$.

Hence, $\quad \frac{A}{G^2} = \frac{a+l}{2} \div al = \frac{a+l}{2al}$.

Therefore, $\quad \frac{A}{G^2} = \frac{1}{H}$, or $\frac{A}{G} = \frac{G}{H}$.

That is, G is a *mean proportional* between A and H. See § 172.

EXERCISES AND PROBLEMS

1. Insert three harmonic means between 22 and 11.

2. Insert six harmonic means between $\frac{1}{8}$ and $\frac{23}{6}$.

3. The first term of a harmonic progression is $\frac{1}{2}$ and the tenth term is $\frac{1}{20}$. Find the intervening terms.

4. Two consecutive terms of a harmonic progression are 5 and 6. Find the next two terms and also the two preceding terms.

5. If a, b, c are in harmonic progression, show that $a \div c = (a - b) \div (b - c)$.

6. Find the arithmetic, geometric, and harmonic means between:

(a) 16 and 36; (b) $m + n$ and $m - n$; (c) $\dfrac{1}{m+n}$ and $\dfrac{1}{m-n}$.

7. The harmonic mean between two numbers exceeds their arithmetic mean by 7, and one number is three times the other. Find the numbers.

8. If x, y, and z are in arithmetic progression, show that mx, my, and mz are also in arithmetic progression.

9. x, y, and z being in harmonic progression, show that $\dfrac{x}{x+y+z}$, $\dfrac{y}{x+y+z}$, and $\dfrac{z}{x+y+z}$ are in harmonic progression, and also that $\dfrac{x}{y+z}$, $\dfrac{y}{x+z}$, and $\dfrac{z}{x+y}$ are in harmonic progression.

10. The sum of three numbers in harmonic progression is 3, and the first is double the third. Find the numbers.

11. The geometric mean between two numbers is $\frac{1}{4}$ and the harmonic mean is $\frac{1}{5}$. Find the numbers.

12. Insert n harmonic means between the numbers a and b.

CHAPTER XIII

THE BINOMIAL FORMULA

221. In Chapter II the following products were obtained:

$$(a+b)^2 = a^2 + 2ab + b^2.$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

By a study of these the following facts may be observed:

1. Each product has one term more than the number of units in the exponent of the binomial.

2. The exponent of a in the *first* term is the same as the exponent of the binomial, and diminishes by unity in each *succeeding* term.
The exponent of b in the *last* term is the same as the exponent of the binomial, and diminishes by unity in each *preceding* term.

3. The sum of the exponents in each term is equal to the exponent of the binomial.

4. The coefficient of the first term is unity; of the second term, the same as the exponent of the binomial; and the coefficient of any other term may be found by multiplying the coefficient of the next preceding term by the exponent of a in that term and dividing this product by a number one greater than the exponent of b in that term.

5. The coefficients of any pair of terms equally distant from the ends are equal.

Statements 2 and 4 form a rule for writing out any power of a binomial up to the fifth. Let us find $(a+b)^6$.

Multiplying $(a+b)^5$ by $a+b$, we have

$$\begin{array}{rl} (a+b)^5(a+b) = & a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\ & \phantom{a^6 +{}} a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\ \hline \text{Hence } (a+b)^6 = & a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \end{array}$$

From this it is seen that the rule holds also for $(a+b)^6$.

PROOF BY MATHEMATICAL INDUCTION

222. A proof that the above rule holds for *all positive integral powers* of a binomial may be made as follows:

First step. Write out the product as it *would be* for the nth power on the supposition that the rule holds.

Then the first term would be a^n and the last term b^n. The second terms from the ends would be $na^{n-1}b$ and nab^{n-1}. The third terms from the ends would be $\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$ and $\frac{n(n-1)}{1\cdot 2}a^2b^{n-2}$. The fourth terms from the ends would be

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 \text{ and } \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3b^{n-3},$$

and so on, giving by the hypothesis,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots + \frac{n(n-1)}{1\cdot 2}a^2b^{n-2} + nab^{n-1} + b^n.$$

Second step. Multiply this expression by $a+b$ and see if the result can be so arranged as to conform to the same rule. Then, $\quad (a+b)^n(a+b)$

$$\begin{array}{l} = a^{n+1} + na^nb + \frac{n(n-1)}{2}a^{n-1}b^2 + \cdots \qquad\qquad + na^2b^{n-1} + ab^n \\ \qquad\quad a^nb \qquad\quad + na^{n-1}b^2 + \cdots + \frac{n(n-1)}{1\cdot 2}a^2b^{n-1} + nab^n + b^{n+1}. \\ \hline \end{array}$$

Hence adding,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^nb + \left[\frac{n(n-1)}{1\cdot 2} + n\right]a^{n-1}b^2 + \cdots$$
$$+ \left[n + \frac{n(n-1)}{1\cdot 2}\right]a^2b^{n-1} + (n+1)ab^n + b^{n+1}.$$

Combining the terms in brackets, we have,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{1\cdot 2}a^{n-1}b^2 + \cdots$$
$$+ \frac{(n+1)n}{1\cdot 2}a^2b^{n-1} + (n+1)ab^n + b^{n+1}.$$

The last result shows that the rule holds for $(a+b)^{n+1}$ *if it holds for* $(a+b)^n$. That is, *if the rule holds for any positive integral exponent, it holds for the next higher integer.*

Third step. It was found above by *actual multiplication* that the rule does hold for $(a+b)^6$. Hence by the above argument we know that the rule holds for $(a+b)^7$.

Moreover, since we now know that the rule holds for $(a+b)^7$, we conclude by the same argument that it holds for $(a+b)^8$, and if for $(a+b)^8$, then for $(a+b)^9$, and so on.

Since this process of extending to higher powers can be carried on indefinitely, we conclude that the five statements in § 221 hold for all positive integral powers of a binomial.

The essence of this proof by **mathematical induction** consists in applying the *supposed* rule to the nth power and finding that the rule does hold for the $(n+1)$th power *if it holds for the nth power.*

223. The general term. According to the rule now known to hold for any positive integral exponent, we may write as many terms of the expansion of $(a+b)^n$ as may be desired, thus:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$$

$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots. \quad \text{I}$$

From this result, called the **binomial formula**, we see:

(1) The exponent of b in any term is one less than the number of that term, and the exponent of a is n minus the exponent of b. Hence the exponent of b in the $(k+1)$st term is k, and that of a is $n-k$.

(2) In the coefficient of any term the last factor in the denominator is the same as the exponent of b in that term, and the last factor in the numerator is one more than the exponent of a.

Hence the $(k+1)$st term, which is called the **general term** is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots k}a^{n-k}b^k. \quad \text{II}$$

224. The process of writing out the power of a binomial is called **expanding the binomial,** and the result is called the **expansion of the binomial.**

Ex. 1. Expand $(x-y)^4$.

In this case $a = x$, $b = -y$, $n = 4$.

Hence substituting in formula I,

$$(x-y)^4 = x^4 + 4x^3(-y) + \frac{4(4-1)}{2}x^2(-y)^2 + \frac{4(4-1)(4-2)}{2\cdot 3}x(-y)^3 + \frac{4(4-1)(4-2)(4-3)}{2\cdot 3\cdot 4}(-y)^4 \qquad (1)$$

$$= x^4 - 4x^3y + \frac{4\cdot 3}{2}x^2y^2 - \frac{4\cdot 3\cdot 2}{2\cdot 3}xy^3 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4}y^4. \qquad (2)$$

Hence $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$ (3)

Notice that this is precisely the same as the expansion of $(x+y)^4$ except that every other term beginning with the second is *negative.*

Ex. 2. Expand $(1-2y)^5$.

Here $a = 1$, $b = -2y$, $n = 5$.

Since the coefficients in the expansion of $(a+b)^5$ are 1, 5, 10, 10, 5, 1, we write at once,

$$(1-2y)^5 = 1^5 + 5\cdot 1^4\cdot(-2y) + 10\cdot 1^3\cdot(-2y)^2 + 10\cdot 1^2\cdot(-2y)^3 + 5\cdot 1\cdot(-2y)^4 + (-2y)^5$$
$$= 1 - 10y + 40y^2 - 80y^3 + 80y^4 - 32y^5.$$

Ex. 3. Expand $\left(\frac{1}{x}+\frac{y}{3}\right)^5$.

Remembering the coefficients just given, we write at once,

$$\left(\frac{1}{x}+\frac{y}{3}\right)^5 = \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(\frac{y}{3}\right) + 10\left(\frac{1}{x}\right)^3\left(\frac{y}{3}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(\frac{y}{3}\right)^3 + 5\left(\frac{1}{x}\right)\left(\frac{y}{3}\right)^4 + \left(\frac{y}{3}\right)^5\left(\frac{1}{x}+\frac{y}{3}\right)^5$$

$$= \frac{1}{x^5} + \frac{5}{3}\frac{y}{x^4} + \frac{10}{9}\frac{y^2}{x^3} + \frac{10}{27}\frac{y^3}{x^2} + \frac{5}{81}\frac{y^4}{x} + \frac{y^5}{243}.$$

In a similar manner any positive integral power of a binomial may be written.

Ex. 4. Write the *sixth term* in the expansion of $(x-2y)^{10}$ without computing any other term.

From II, § 223, we know the $(k+1)$st term for the nth power of $a+b$, namely,

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{2\cdot 3\cdot 4\cdots k}a^{n-k}b^k.$$

In this case $a=x$, $b=-2y$, $n=10$, $k+1=6$. Hence $k=5$.

Substituting these particular values, we have

$$\frac{10(10-1)(10-2)\cdots(10-5+1)}{2\cdot 3\cdot 4\cdot 5}x^{10-5}(-2y)^5$$

$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2\cdot 3\cdot 4\cdot 5}x^5(-32y^5)$$

$$=-32\cdot 252\,x^5y^5=-8064\,x^5y^5.$$

EXERCISES

1. Make a list of the coefficients for each power of a binomial from the 2d to the 10th.

Expand the following:

2. $(x-y)^3$.
3. $(2x+3)^3$.
4. $(3x+2y)^4$.
5. $(3+y)^5$.
6. $(x^3+y)^6$.
7. $(x-y^3)^6$.
8. $(x^3-y^2)^7$.
9. $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^4$.
10. $(x^{-1}+y^{-2})^5$.
11. $(a-b)^8$.
12. $(x+y)^9$.
13. $(m-n)^{10}$.
14. $(r^{\frac{1}{2}}+s^2)^4$.
15. $(c^{-2}-d^{-\frac{1}{2}})^5$.
16. $(\sqrt{a}-\sqrt{b})^6$.
17. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^3$.
18. $\left(\frac{2x}{y^2}-y\sqrt{x}\right)^3$.
19. $\left(\frac{\sqrt{m}}{\sqrt[3]{n^2}}+\sqrt{\frac{y}{n}}\right)^4$.
20. $\left(\frac{c\sqrt[3]{c}}{\sqrt[5]{d^4}}-\frac{\sqrt[3]{d}}{c}\right)^7$.
21. $(2a^2x^{-2}-3by^{-3})^4$.
22. $(3xy^{-3}-x^{-3}y)^8$.

In each of the following find the term called for without finding any other term:

23. The 5th term of $(a+b)^{12}$.

24. The 7th term of $(3x-2y)^{11}$.

25. The 6th term of $(\sqrt{x}-\sqrt[3]{y})^{10}$.

26. The 9th term of $(x-y)^{25}$.

27. The 8th term of $(\frac{1}{2}m-\frac{1}{3}n)^{18}$.

28. The 7th term of $(a^2b-ab^2)^{30}$.

29. The 6th term of $(a-a^{-1})^{2k}$.

30. The 11th term of $(x^2y-x^{-2}y^{-1})^{3m}$.

31. The 5th term from each end of the expansion of $(a-b)^{20}$.

32. The 7th term from each end of $(a\sqrt{a}-b\sqrt{b})^{21}$.

33. Which term, counting from the beginning, has the same coefficient as the 7th term of $(a+b)^{10}$? Verify by finding both coefficients. How do the exponents differ in these terms?

34. What other term has the same coefficient as the 19th term of $(a+b)^{24}$? How do the exponents differ? Find in the shortest way the 21st term of $(a+b)^{25}$.

35. Find the 87th term of $(a+b)^{90}$.

36. Find the 53d term of $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^{56}$.

37. What other term has the same coefficient as the 5th term in the expansion of $(x+y)^{19}$?

38. Expand $[(a+b)+c]^3$ by the binomial formula.

39. Expand $[1+(2x+3y)]^4$ by the binomial formula.

40. Expand $(2x-3y+4z)^3$ by the binomial formula.

41. Write the $(k+1)$st term of $(a+b)^n$. Write the $(n+1)$st term of $(a+b)^n$. Show that the next and also all succeeding terms after the $(n+1)$st term have zero coefficients, thus proving that there are exactly $n+1$ terms in the expansion.